Discussion of the Supplementary Lectures

The suggested format is to have 1 section a week be devoted to a lecture on a topic that supplements the usual lectures. The attached syllabus gives a set of suggested topics. Assuming that 2 midterms are given, the course naturally falls into 3 parts.

The first 3 supplementary lectures should be devoted to helping the students learn very basic proof techniques. Thus in presenting the basic material in Appendices C and D, the terms of basic logic should be explained e.g. contrapositive, converse, implication, negation, etc. In the 3rd lecture, proof by induction should be covered. If the instructor gives the replacement theorem in the form given in the book, it provides a good advanced example of induction and the TA could devote most of that lecture to going over its proof in complete detail.

After the first midterm, it will help the students to review matrix computations. Here the TA may refer to Chapters 3 and 4 and review row reduction to compute rank and inverses. It is advised that 3 X 3 examples be worked out. Also, determinant calculations could be reviewed. Then an in-depth application of diagonalization, for example the computation of higher powers of a matrix can be presented. The TA (with the instructor's approval) should present a topic of interest, say from Sec. 5.3.

After the second midterm, more detailed examples could be discussed. A treatment of the inner product space H defined on p. 332 and a review of properties of integrals could be given. To supplement Sec. 6.3, the TA could give a lecture on least squares from p. 360. Another possible topic is minimal solutions to systems of linear equations on p. 364. The last lecture before the final could be used as a review session.

Possible Syllabus for Supplementary Lectures by the Teaching Assistant

Lecture	**Text References**	**Topics**
1	App.C. p 552	Field Axioms, Thms C.1, C.2 Emphasis on Logical Reasoning in proofs
2	App.D.	Complex Numbers Continue illustration of basic techniques of proofs
3	.	Proof by induction - elementary examples as well as a detailed review of the proof of the replacement theorem - Thm. 1.10
4	3.1, 3.2 Selections	Review the mechanics of matrix calculations - row reduction, rank, inverse, 3 X 3 examples
5	4.1-4.3 Selections	Calculation of determinants
6	.	Detailed discussion of an example involving high powers of a matrix - e.g. choose material from 5.3 or use another example, e.g. linear recurrences
7	.	Continue previous example and/or review for 2nd midterm
8	.	Discussion of the inner product space H defined on p. 332 - review of needed properties of integrals
9	.	Least squares Approx. p. 360 or Minimal Solutions example p. 364
10 (if given)	.	Review for Final

Possible Syllabus for Supplementary Lectures by the Teaching Assistant

Lecture 1: Fields. (This could include a proof that $a^2 = 2b^2$ has no nonzero solutions in integers and consequently no solutions in rationals either, as a consequence, the real numbers of the form $\mathrm{Q}[\sqrt{2}]$ form a field.)
Lecture 2: Proof by induction. The examples of proof should be related to linear algebra. (Thus the familiar example $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$ should not be used.) One possible proposition would be: If A is on a upper triangular matrix with diagonal entries $d_1, \dots, d_n$ then
 i) $\det(A - tI_n) = (d_1 - t)\dots(d_n - t)$.
 ii) $(A - d_1I_n)(A - d_2I_n)\dots(A - d_nI_n) = 0$ in $M_{n\times n}(F)$.
Students come into 115A knowing enough about determinants to follow the proof of i).
Lecture 3: One lecture on logic. The meaning of implication, of the converse, and contrapositive of a statement, and the negation of statements. For example, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ find the matrix representative of $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ defined by $T(X) = AXB$, relative to the standard basis for $M_{2\times 2}(F)$.
Lecture 4: Review row reduction and computation of the rank of matrices.
Lecture 5: Review matrix inverses and the proof that, for $n \times n$ matrices A and B $\det(AB) = \det(A)\det(B)$. The fact will be familiar but the proof will not.
Lecture 6: Supplementary material in Section 5.2, consisting of p. 272-275 and Exercise 15. Some instructors may wish to cover part of Section 5.3 instead.
Lecture 7: Discussion of the matrix exponential.
Lecture 8: Orthogonal polynomials. This lecture might have the following content:
 i) Given a positive weight function $r(x)$ on $[a, b]$, the Gram-Schmidt process yields a sequence $\{R_n(x)\}$ of orthogonal polynomials. Then the polynomials always satisfy a 3 term recurrence relation

$$R_{n+1}(x) = (x - a_n)R_n(x) + b_nR_{n-1}(x).$$

 ii) Gram-Schmidt is extremely tedious as a method of orthogonalizing the sequence $\{1, x, x^2, x^3, \dots\}$ on $[-1, 1]$.
Defining

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2 - 1)^n$$

we see that the Legendre polynomials are orthogonal on $[-1, 1]$, by an inductive argument reinforcing the importance of induction as a means of proof.
 iii) Assuming the Lagrange interpolation polynomials have been discussed earlier, we let $r_1 < r_2 < \dots < r_{n+1}$ be the roots of P_{n+1}. We assume that they are real and lie in $(-1, 1)$. Letting $f^{(i)}(x)$ be the Lagrange interpolation polynomials for $1 \le i \le n + 1$, let $k_i = \int_{-1}^{1} f^{(i)}(x)dx$. Then for f a polynomial of degree at most n,

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n+1} f(r_i)k_i$$

and this quadrature method remains exact for polynomials of degree $\le 2n + 1$. (Gaussian quadrature.)
(Remark: The department has discussed the tendency of students to forget material from their lower division courses immediately after the final exam. I am suggesting this content for Lecture 8 because it does integrate ideas from calculus and from linear algebra. If the TA or instructor prefers that less be covered in Lecture 8, i) and ii) could be included, iii) omitted, and an additional problem from p. 353-356 discussed.
Lecture 9: Review or leeway.
Lecture 10: An instructive application of simultaneous diagonalization is to rank one perturbations. For example, the matrix

$$\begin{pmatrix} t + a^2 & ab & ac \\ ab & t + b^2 & bc \\ ac & bc & t + c^2 \end{pmatrix}$$

has eigenvalues $t, t, t + a^2 + b^2 + c^2$, and similarly for an $n \times n$ matrix. Finally, if time allows, one proves: If A is a symmetric operator on R^n with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, and V is an $(n-1)$-dimensional subspace of R^n, the quadratic form $q(x) = \langle A_x, x\rangle$ (where $x \in V$) can be diagonalized with eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ satisfying $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$.
TA and instructor might prefer to skip the example of rank one perturbations.
I consider that this material fits into the tenth week better than the ninth week.