Perspective on Geometric Analysis:

Geometric Structures

III

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Talk in UCLA

When the topological method of surgery and gluing fails, we have to find a method that does not depend on the detailed topological information of the manifold. The best example is the proof of the Severi conjecture and the Poincarè conjecture and the geometrization conjecture.

The beauty of the method of nonlinear different equation is that we can keep on deforming some unknown structure until we can recognize them eventually. The control on this process of deformation depends on careful a priori estimate of the nonlinear equation. However, if the structure is to be changed in a large scale, standard energy estimate usually cannot be used as the underliving Sobolov inequality depends on the geometric structure and need not hold in general. Hence maximum principle is used in most cases.

A fruitful idea to construct geometric structure is to construct metrics that satisfy the Einstein equation. We demand that the Ricci tensor of the metric be proportional to the metric itself. This can be considered as a generalization of the Poincarè metric to higher dimensions. This is an elliptic system, if we identify metrics up to diffeomorphisms. The problem of existence of an Einstein metric is really a very difficult but central problem in geometry.

One can obtain such metrics by a variational principle: After normalization of the metrics by setting their volume equal to one, we minimize the total scalar curvature in each fixed conformal class; then we vary the conformal class and maximize the (constant) scalar curvature. The first part is called the Yamabe problem and was settled by the works of Trudinger-Aubin-Schoen.

The most subtle part was the case when the manifold is conformally flat, where Schoen made use of the positive mass conjecture to control the Green's function of the conformally invariant operator and hence settle this famous analytic problem. The relation of this problem with general relativity is a pleasant surprise and should be considered as an important development in geometric analysis. The second part of maximization among all conformal structure is much more difficult. Schoen and his students, and also M. Anderson have made contributions towards this approach.

Let me now discuss the other two major general approaches to constructing Einstein-metrics. The first one is to solve the equation on a space with certain internal symmetries. For such manifolds, the ability to choose a special gauge, such as holomorphic coordinates is very helpful. The space with internal symmetry can be a Kähler manifold or a manifold with special holonomy group.

A very important example is given by the Calabi conjecture, where one asked whether the necessary condition for the first Chern class to have definite sign is also sufficient for the existence of Kähler-Einstein metric.

Algebraic varieties are classified according to the map of the manifold into the complex projective space by powers of the canonical line bundle. If the map is an immersion at generic point, the manifold is called an algebraic manifold of general type. This class of manifolds comprises the majority of algebraic manifolds, and these manifolds can be considered as generalizations of algebraic curves of higher genus.

In general, the above canonical map may have a "base point" and hence be singular. However, the minimal model theory of the Italian and Japanese school (Castelnuovo, Fano, Enriques, Severi, Bombieri, Kodaira, Mori, Kawamata, Miyaoka, Inoue) showed that an algebraic manifold of general type can be contracted to a certain minimal model, where the canonical map has no base point. In this case, the first Chern class of the minimal model is non-positive and negative in a Zariski open set.

Most algebraic manifolds of general type have negative first Chern class. In this case, Aubin and I independently proved the existence and uniqueness of a Kähler-Einstein metric.

For the general case of minimal models of manifolds of general type, the first Chern class is not negative everywhere. Hence it does not admit a regular Kähler-Einstein metric.

However, it admits a canonical Kähler-Einstein metric which may have singularities. This statement was observed by me right after I wrote my paper on the Calabi conjecture, where I also discussed the regularity of degenerate Kähler-Einstein metrics. (Tsuji later reproved this theorem in 1985 using Hamilton's Ricci flow.)

The singularity of this canonical Kähler-Einstein metric that I constructed on manifolds of general type is not so easy to handle. By making some assumption on the divisors, Cheng-Yau and later Tian-Yau contributed to understanding of the structure of these metrics. These metrics give important algebraic geometric informations of the manifolds.

In 1976, I observed that the Kähler-Einstein metric can be used to settle important questions in algebraic geometry. An important contribution is the algebraic-geometric characterization of Shimura varieties: quotients of Hermitian symmetric domains by discrete groups. They are characterized by the statement that certain natural bundle, constructed from tensor product of tangent bundles, has nontrivial holomorphic section.

The other important assertions are the inequalities between Chern numbers for algebraic manifolds. For an algebraic surface, I proved $3C_2(M) \geq C_1^2(M)$, an inequality which was independently proved by Miyaoka by algebraic means. I proved further that equality holds only if M has constant holomorphic sectional curvature.

It is the last assertion that enabled me to prove that there is only one complex structure on the complex projective plane. This statement was a famous conjecture of Severi.

The construction of Ricci flat Kähler metric has been used extensively in both algebraic geometry and string theory, such as Torelli theorem for $K3$ surfaces and deformation of complex structure.

The construction of Kähler-Einstein metric with positive scalar curvature has been a very active field. In early eighties, I proposed its existence in relation to stability of the manifolds.

In the hands of Donaldson, and others, we see that my proposal is close to be realized. It gives new informations about the algebraic geometric stability of manifolds.

In general, there should be an interesting program to study Kähler-Einstein metrics on the moduli space of either complex structures or stable bundles. It should provide some informations for the moduli space. For example, recently, using this metric, Liu-Sun-Yau proved the Mumford stability of the logarithmic cotangent bundle of the moduli spaces of Riemann surfaces.

Hence we see that by constructing new geometric structure through nonlinear partial differential equation, one can solve problems in algebraic geometry that are a priori independent of this new geometric structure.

A holomorphic coordinate system is a very nice gauge and a Kähler metric is a beautiful metric as it depends only on one function. When we come to the space of Riemannian metrics, we need to understand a large system of nonlinear equations invariant under the group of diffeomorphism. The choice of gauge causes difficulty.

The Severi conjecture can be considered as a complex analog of the Poincarè conjecture. The fact that Einstein metrics were useful in setting the Severi conjecture indicates that these metrics should also be useful for the geometrization conjecture and hence the Poincarè conjecture. This was what we believed in the late seventies.

Many methods motivated by the calculus of variation were proposed. The most promising method was due to Hamilton who proposed to deform any metric along the negative of its Ricci curvature. The development of the Ricci flow has gone through several important stages of development.

The first decisive one was Hamilton's demonstration of the global convergence of the Ricci flow (1982) when the initial metric has positive Ricci curvature. This is a fundamental contribution that give confidence on the importance of the equation.

To move further, it was immediately clear that one needs to control the singularities of the flow. This was studied extensively by Hamilton. The necessary a priori estimate was based on Hamilton's spectacular generalization of the works of Li-Yau (1984).

Li-Yau introduced a distance function on space-time to control the precise behavior of the parabolic system near the singularity. The concept appears naturally from the point of view of a priori estimate. For example, if the equation is

$$\frac{\partial u}{\partial t} = \triangle u - Vu.$$

The distance introduced by Li-Yau is given by

$$\begin{aligned}
&d((x,t_1),(y,t_2)) \\
&= \inf_r \Big\{ \frac{1}{4(t_2-t_1)} \int_0^1 |\, \dot{r}\, |^2 \\
&+ (t_2-t_1)\int_0^1 V\Big(r(s),(1-s)t_2+st_1\Big)\Big\}.
\end{aligned}$$

The kernel of the parabolic equation can then be estimated by this distance function.

The potential V is naturally replaced by the scalar curvature in the case of Ricci flow as it appears in the action of gravity. This is what Perelman did later. The idea of Li-Yau-Hamilton come from the careful study of maximum principle. The basic philosophy of LYH is to study the extreme situation. In the case of Ricci flow, one looks at the soliton solution and verify some equality holds along the soliton and such equality can be turned to be estimates for general solutions of the parabolic system, via maximum principle.

In the nineties, Hamilton was able to classify singularities of the Ricci flow in three dimension and prove the geometrization conjecture if the curvature of the flow is uniformly bounded. These are very deep works both from the point of view of geometry and analysis. Many ideas in geometric analysis were used. This includes the proof of the positive mass conjecture, the injectivity radius estimate and an improved version of the Mostow rigidity theorem. In particular, he introduced the concept of Ricci flow with surgery.

In his classification of singularities, Hamilton could not determine the existence or nonexistence of one type of singularity which he called cigar. This type of singularity was proved to be non-existent by Perelman in 2002 in an elegant manner. Perelman then extended the work of Hamilton on flows with surgery. Among many creative ideas, he found a priori estimates for the gradient of the scalar curvature, the concept of reduced volume and a new way to perform surgery with control.

The accumulated works of Hamilton-Perelman are spectacular. Today, 5 years after the first preprint of Perelman was available, several groups of mathematicians have put forward their manuscripts explaining their understandings on how Hamilton-Perelman's ideas can be put together to prove the Poincare conjecture; at the same time, other experts are still working diligently on the proof of this century old conjecture.

Besides the Poincarè conjecture, Ricci flow has many other applications: A very important one is the contribution due to Chau, Chen, Ni, Tam and Zhu, towards the proof of the conjecture that every complete noncompact Kähler manifold with positive bisectional curvature is bi-holomorphic to $\mathbb{C}^n$. (I made this conjecture in 1972 as a generalization to higher dimension of the uniformization theorem. Proceeding to the conjecture, there were important works of Greene-Wu to proving Steiness of the complete noncompact Kähler manifold with positive sectional curvature.)

More recently, several old problems were solved by using the classical results of Hamilton that were published in 1983, 1986 and 1997.

The most outstanding one is the recent result of Brendle-Schoen; they proved that manifolds with pointwise quarter-pinching curvature are diffeomorphic to manifolds with constant positive curvature.

This question has puzzled mathematicians for more than half a century.

It has been studied by many experts in differential geometry.

Back in 1950s, Rauch was the first one who introduce the concept of pinching condition. Berger and Klingenberg proved such a manifold to be homeomorphic to a sphere when it is simply connected.

The diffeomorphic type of the manifolds is far more difficult to understand. For example, Gromoll's thesis achieved a partial result toward settling such a result: he assumed a much stronger pinching condition.

The result of Brendle-Schoen achieved optimal pinching condition. More remarkably, they only need pointwise pinching condition and do not have to assume simply-connectivity. Both of these conditions are not accessible by the older methods of comparison theorems.

This result partially builds on fundamental work by Böhm and Wilking who proved that a manifold with positive curvature operator is diffeomorphic to a spherical space form.

Therefore, the program on Ricci flow laid down by Hamilton in 1983 has opened a new era for geometric analysts to build geometric structures.

Other obvious problems are to construct geometric structure on other low dimensional manifolds, especially four-dimensional manifolds. Besides the fundamental works based

The Atiyah-Singer index formula gives very important obstructions for the existence of integrable complex structures on surfaces, as was found by Kodaira. The moduli spaces of holomorphic vector bundles have been a major source for Donaldson to provide invariants for smooth structures. On the other hand, the existence of pseudoholomorphic curves based on Seiberg-Witten invariant constructed by Taubes is a powerful tool for symplectic topology. It seems natural that one should build geometric structures over a smooth manifold that include all these types of information.

The integrability condition derived from Atiyah-Singer formula for almost complex structures in $\dim_{\mathbb{C}} \geq 3$ is not powerful enough to rule out the following conjecture:

For $\dim_{\mathbb{C}} \geq 3$, every almost complex manifold admits an integrable complex structure.

If this conjecture is true, we need to build geometry over such nonKähler complex manifolds. This is especially interesting in higher dimension. It is possible to deform an algebraic manifold to another one with different topology by tunneling through nonKähler structures. A good example is related to the Clemens-Friedman construction that one can collapse rational curves in a Calabi-Yau three manifold to conifold singularity. Then by smoothing the singularity, one obtains nonsingular nonKähler manifold. Reversing the procedure, one may get another Calabi-Yau manifold.

Reid proposed that this procedure may connect all Calabi-Yau manifolds in three dimension. There is perhaps no reason to restrict ourselves only to Calabi-Yau manifolds, but to more general algebraic manifolds on a fixed topological manifold, is there other general construction to deform one algebraic structure from one component of the moduli space of a complex structure to other component through nonKähler complex structures?

Non-Kähler complex structures are difficult to handle geometrically. However, there is an interesting concept of Hermitian structure that can be useful. This is the class of balanced structure.

An Hermitian metric ω is called balanced iff

$$d(\omega^{n-1}) = 0.$$

It was first studied by M. Michelsohn and Alessandrini-Bassanelli, who observed that twistor space admits a balanced metric and that existence of balanced metric is invariant under birational transformation. Recently, it came up in the theory of Heterotic string, based on a warped product compactification.

Strominger suggests that there should be a holomorphic vector bundle that should admit a Hermitian Yang-Mills connection and that there should be Hermitian metric that is conformally balanced. To be more precise, there should be a holomorphic 3−form Ω so that

$$d(\| \Omega \|_\omega \omega^2) = 0,$$

where ω is the Hermitian form. An important link between the bundle and the metric is that connections on both structures give trivial first Chern form and the difference between their second Chern forms can be written as $\sqrt{-1}\partial\bar{\partial}\omega$.

This geometric structure constructed for Heterotic string theory is based on construction of parallel spinors and the anomaly equation required by quantization of string theory.

On the other hand, general existence theorem for the Strominger system is still not known.

An interesting mathematical question is to construct a balanced complex three manifold with a nonvanishing holomorphic 3-form. Then we like to construct a stable holomorphic vector bundle that satisfies all of the above equations of Strominger.

Jun Li and I proved the existence of Strominger system by perturbing around the Calabi-Yau metric.

The first example on a nonKähler manifold is due to Fu-Yau. It is obtained by forming a torus fiber bundle over $K3$ surface (due to Dasgupta-Rajesh-Sethi, Becker-Becker-Dasgupta-Green and Goldstein-Prokushkin).

The construction of Strominger system over this manifold can be achieved if we can solve the following complex Monge-Ampere equation:

$$\Delta(e^u - \frac{\alpha'}{2} f e^{-u}) + 4\alpha' \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0,$$

where f and μ are given functions on $K3$ surface S so that $f \geq 0$ and $\int_S \mu = 0$. This was achieved by Fu-Yau based on a priori estimates of u, which is more complicated than those used in Calabi conjecture.

There are nontrivial interpretation of the Fu-Yau example through conformal field theory.

The supersymmetric heterotic string gives an $SU(3)$ Hermitian connection on the tangent bundle. But the connection has torsion (which is trace free).

If we are interested in G-structure on the tangent bundle with $G \subseteq U(n)$, it can be accomplished by considering the work of Donaldson-Uhlenbeck-Yau on stable holomorphic bundles over Kähler manifold. The work generalizes the work of Narasimhan-Seshadri for algebraic curves.

It was extended to nonKähler manifolds by Li-Yau where the base complex manifold admit a Gauduchon metric ω with

$$\partial\bar{\partial}(\omega^{n-1}) = 0.$$

If the tangent bundle T is stable and if some irreducible subbundles constructed from tensor product of T admits nontrivial holomorphic section, the structure group can be reduced. The major question is how to control the torsion of this connection by choosing ω suitably.

In the other direction, one should mention that Smith-Thomas-Yau succeeded to construct symplectic manifold mirror to the Clemens-Friedman construction. While the Clemens-Friedman construction leads to nonKähler complex structures over connected sums $S^3 \times S^3$, the Smith-Thomas-Yau construction lead to symplectic non-complex structure over connected sums of $\mathbb{C}P^3$ (which may not admit any integrable complex structure).

We expect a mirror structure for the Strominger system in symplectic geometry, where we hope to build an almost complex structure compatible to the symplectic form. They should satisfy a good system of equations. We expect that special Lagrangian cycles and pseudoholomorphic curves will play roles in such a new structure which is dual to the above system of equations of Strominger.

The inspirations from string theory has given amazingly deep insight into the structure of Calabi-Yau manifolds which are manifolds with holonomy group $SU(n)$.

Constructions of geometric structures by coupling metrics with vector bundles and submanifolds should give a new direction in geometry, as they may exhibit supersymmetry. An important idea provided by string theory is that duality exists between supersymmetric manifolds. Duality allows us to compute difficult geometric information by perturbation methods on the dual objects.

General relativity and string theory have inspired a great deal of geometric ideas and it has been very fruitful. Nature also tells us everything vibrates and there should be intrinsic frequency associated to our geometric structure. In the classical geometry, we have an elliptic operator associated to deformation of the structure. For space of Einstein metrics, it is called the Lichnerowicz operator. It will be interesting to study the spectrum of this operator.

Quantum gravity may provide a deeper concept. A successful construction of quantum geometry will change our scope of geometric structures.

Einstein (Herbert Spencer lecture at Oxford in 1933):

Pure logical thinking cannot yield us any knowledge of the empirical world. All knowledge of reality starts from experience and ends in it. Propositions arrived at by purely logical means are completely empty as regards reality.