Perspective on Geometric Analysis:

Geometric Structures

II

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The concept of geometric structure has been enriched continuously. It has been found that metrics with special holonomy group may not be enough to describe the structure. In order to explain this, I will motivate the idea through the concept of duality in string theory. Let us start with some classical examples.

The theory of Lie groups and their discrete subgroups gives rise to Cartan's theory of locally symmetric and homogeneous spaces. They provide examples with rich properties for geometers and analysts. many important properties of these spaces were obtained when we consider them to be moduli space of other geometric objects.

For example, the Siegel upper space can be considered as moduli space of abelian varieties. Occasionally, moduli space of some algebraic manifolds can be locally Hermitian symmetric: Such manifolds include $K3$-surfaces, Calabi-Yau manifolds obtained by taking branched cover over $\mathbb{C}P^3$ along eight hyperplanes or cubic surfaces. Many hyperKähler manifolds such as symmetric products of $K3$ surfaces can be considered as moduli space of semi-stable vector bundles over hyperKähler manifolds.

On the other hand, to understand geometric structures, it is important to understand non-linear transformations between these spaces that are of geometric importance. For example, if H and K are two closed subgroup in a Lie group G, one can construct a natural map from sheaves or cohomology classes of the space G/H to the space G/K by pulling back the objects from G/H to $G/(H \cap K)$. After twisting by some universal object on $G/(H \cap K)$, one can push the product to objects on G/K:

$$\begin{array}{ccc} & G/(H \cap K) & \\ \swarrow & & \searrow \\ G/H & & G/K \end{array}$$

As was observed by Chern, the classical Kinematic Formulae of Poincaré, Santalo and Blaschke can be formulated in terms of the above transformation by taking G to be the group of motions on the homogeneous space where incidence relations of submanifolds are considered.

This kind of transformations also appeared in many places. A very important one is the case of four dimensional manifold M and we consider the moduli space $\mathcal{M}$ of rank two bundles over M whose curvature is self-dual. On the product space $M \times \mathcal{M}$, there is a rank two universal bundle V and we can use the second Chern class of V to transform second cohomology of M to $\mathcal{M}$ and obtain the Donaldson polynomials.

Another important case is the T-duality that has played an important role in number theory and algebraic geometry.

Let $T^n = \mathbb{R}^n/\mathbb{Z}_n$ be a torus and $(T^n)^* = \mathbb{R}^n/(\mathbb{Z}_n)^*$ be the dual torus, which can be considered as the moduli space of complex flat line bundles over T^n. Then we have the following diagram

$$\begin{array}{ccc} & L & \\ & \downarrow & \\ & T^n \times (T^n)^* & \\ \swarrow & & \searrow \\ T^n & & (T^n)^* \end{array}$$

There is a universal complex line bundle L over $T^n \times (T^n)^*$ so that L restricted to $T^n \times \{q\}$ is isomorphic to q. We can pull back cohomology classes from T^n to $T^n \times (T^n)^*$ where we multiply the class by $\exp(c_1(L))$. Then we can push the product class to $(T^n)^*$. Such a transform can be considered as a nonlinear transform between the torus T^n and its dual $(T^n)^*$. It is called $T-$duality in the recent developments in string theory.

Note that when $n = 1$, this is the duality between circle of radius r to circle of radius $\frac{1}{r}$.

Strominger-Yau-Zaslow (1995) found that a certain algebraic manifold M (Calabi-Yau) admits a T^3 fiber structure over S^3 where generic fibers are T^3. By replacing T^3 by $(T^3)^*$, we obtain another algebraic manifold M^* which is also Calabi-Yau.

By performing a family version of the above $T-$duality and a Legendre transformation on some affine structures on the base, one obtains a transform that maps one geometric structure over the algebraic manifold M to another geometric structure over M^*. (The affine structure on the base space is the one described previously where we have a potential for the metric and the Legendre transform acts on those potentials.)

Note M and M^* may be topologically distinct. This transformation has many important properties. For example, holomorphic bundles V over M are supposed to be mapped to special Lagrangian cycles C in M^*.

In terms of cohomology, the class $Ch(V)\sqrt{Tod(M)}$ in $H^0(M)\oplus H^2(M)\oplus H^4(M)\oplus H^6(M)$ is mapped to cohomology class of $[C]$ in $H^3(M^*)$. The fact that the algebraic bundles of M are mapped to $H^3(M^*)$ raise the following question:

If M and M^* are defined over some number field, will the Frobenius action on the Étale cohomology of $H^3(M)$ be mapped to certain action on the K groups defined by algebraic vector bundles? Will the Adams operation play a role?

SYZ argued that the above nonlinear transform is the same as the mysterious mirror symmetry that was initiated by Greene-Plesser, Candelas-de la Ossa-Green-Parkes based on speculations of conformal field theory.

Both the analytic and algebraic properties of the mirror transform are spectacular. However, they are not yet well-understood.

On the other hand, it has already produced a powerful method in geometry. For example, it allows algebraic geometers to calculate the number of algebraic curves in a Calabi-Yau manifold. This was a major classical problem in algebraic geometry. It was solved by Candelas et. al., in that they found the right formula. The rigorous mathematical proof came from the works of Liu-Lian-Yau and Givental.

In principle, we can extend the above T-duality to a more general situation. For example, T^n can be replaced by a $K3$-surface or other algebraic manifold and $(T^n)^*$ can be replaced by the moduli space of semi-stable holomorphic bundles over that manifold. In this case, L can be replaced by the universal bundle. Gukov-Yau-Zaslow observed that certain manifolds with holonomy group G_2 have a fiber structure with fiber given by $K3$ surfaces and they are dual to algebraic manifolds which are Calabi-Yau.

The arguments of SYZ and GYZ are based on brane theory, a quantized version of string theory. The belief that the transformation should work well for fibrations with singularities comes from intuition that arose from physics. Mirror symmetry gives rise to many conjectures in geometry which were proved later by rigorous mathematics. The mathematical proof in turn justifies the intuition of the physicists.

Let us now examine how submanifolds can help the construction of geometric structures. It has been an open problem in geometry to construct an explicit metric on a $K3$-surface with holonomy group $SU(2)$. Greene, Shapere, Vafa, and I found an explicit metric (with $SU(2)$ holonomy) on the $K3$-surface fibered over the two sphere with torus fiber. All the fibers have flat metric.

However, our metric is singular along the singular fiber. It is believed that one can perturb this singular metric to be a smooth one with $SU(2)$ holonomy. The perturbation series is believed to be expressible in terms of areas of holomorphic disks with boundary specified to be a subset of the fiber torus. The motivation comes from the interpretation of our metric as a semi-classical approximation to the quantum theory based on the $K3$ surface. The holomorphic disks are instanton corrections.

There is a similar picture for three-dimensional Calabi-Yau manifolds.

In the process of performing the mirror transform, the metric and the complex structure is perturbed by quantities that come from holomorphic cycles or bundles. Hence, it is reasonable to believe that a good geometric structure should include a metric with a certain holonomy group, a space of bundles that have special holonomy group, and a space of cycles such as holomorphic cycles or special Lagrangian cycles. (The Lagrangian that appeared in low energy string theory includes all these quantities and some scalar functions.)

Philosophically, we know that certain subspace of functions can determine the space where they are defined. In fact, algebraic geometers use the rings of algebraic functions to determine the algebraic structure of the manifold. Analytically, we can use solutions of differential equations constructed from the metric to determine the geometric structure.

Obvious functions are harmonic functions, eigenfunctions, eigenforms or spinors. But there are many naturally defined nonlinear differential operators such as the Monge-Amperè operator. Solutions of these nonlinear operators can be directly related to the construction of the metric.

The moduli space of self-dual Yang-Mills bundles or Seiberg-Witten equations have been used by Donaldson et. al. to detect the topological structure of the manifold. One expects that more refined properties of geometric structures can be determined by special bundles or special cycles.

Intuitions from physics have been very useful. In fact, an ultimate goal of geometry is to find a geometric structure that can describe quantum physics when distance is small and general relativity when distance is large. For such a picture, the classical view of spacetime is expected to be changed drastically.

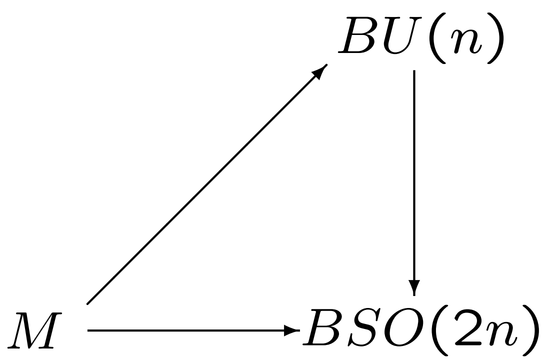
Classical relativity has been verified successfully. The large scale structure of spacetime is therefore in reasonable good shape. However, curvature (or gravity) can drive spacetime to form singularities, which may have to be understood and resolved by quantum physics. The famous conjecture of Penrose says that generic singularity in classical relativity has to be of black hole type.

Singularities are places where physical laws do not hold. What it means is that classical concept of spacetime is not adequate to describe physics at small scale. For small scale structure of spacetime, quantum field theory has to be brought in and it is likely that all the quantities such as bundles and cycles will contribute.

Let me now discuss the approach from the point of view of geometric analysis to construct geometric structures. Two major ways had been developed: one is by gluing structures together and the other one is by calculus of variation or deformation by parabolic equations.

Given a smooth manifold, how does one construct geometric structures over such a manifold? Ideally we would like to find necessary and sufficient conditions in terms of algebraic topological data such as homology classes, homotopy groups and characteristic classes of the manifold.

This is indeed possible for questions such as the existence of almost complex structure by studying the classifying map of the manifold into the classifying space $BU(n)$. The question is reduced to study the lifting of the map to $B(SO(2n))$ which classifies the tangent bundle to a map into $BU(n)$. It is a homotopic question and is completely understood when $n \leq 4$.



In principle, we can replace $U(n)$ by other Lie subgroups of the orthogonal group in the above discussion.

It would be useful to find a necessary and sufficient condition for the existence of G_2-structure on a seven dimensional manifold where the associated three form is closed.

The question of existence of geometric structures is very much related to uniqueness. One can of course relax uniqueness to finite dimensionality of the geometric structures. Only in such cases, techniques of elliptic or parabolic theory of differential equations can be useful. Fortunately, most of the geometric structures have this finite dimensionality property.

However, it should be pointed out that there can be infinitely many distinct components of complex structures on a fixed compact manifold. It will be useful to classify all the possible Chern classes of such complex structures. Similarly, there may be infinite number of components of symplectic structures on a given compact manifold, all of whose symplectic forms belong to the same cohomology class.

The most direct way to construct geometric structures is to perform surgery on manifolds: replacing one handlebody by another handlebody. In the process, one needs to make sure that the new handlebody has compatible geometric structure and the gluing is smooth. The detail of the geometric structure on a manifold with boundary is thus important.

A beautiful example is Thurston's approach to constructing hyperbolic metrics on atoroidal irreducible three-manifolds. Thurston found an important generalization of the rigidity theorem of Mostow on hyperbolic manifolds to three-manifolds with geodesic boundary. The hyperbolic structure is determined by its fundamental group and the conformal structure on the boundary. The possibility of gluing two such manifolds is obtained by a fixed point formula on the Teichmüller space.

Another example is given by the work of Schoen-Yau and Gromov-Lawson on the classification of manifolds with positive scalar curvature. They prove that surgery on embedded spheres with codimension ≥ 3 preserves the existence of metrics with positive scalar curvature.

Construction of geometric structures on a manifold by surgery can be powerful, as many tools of algebraic topology can be brought in. However, the gluing procedure usually involves some question of convexity. For example, a ball is convex for most geometric structures, and in order to glue it to another manifold along the boundary, the boundary of the other manifold has to be concave in a suitable manner. However, in conformal geometry, inversion turns the ball inside out. Therefore, one can prove that the connected sum of conformally flat manifolds is still conformally flat.

It is much more difficult to glue complex manifolds along a complex submanifold unless the normal bundle of the submanifold is trivial. Even in such cases, it remains to find obstructions to constructing an integrable complex structure on the connected sum of two complex manifolds along the complex submanifold. (If the normal bundle of the complex submanifold is negative, one can perform a contraction and a suitable surgery can be carried out.)

The idea of combining methods from geometric analysis and gluing a geometric structure to a given manifold was initiated by the pioneering work of Taubes. He was the first one to construct anti-self dual bundles on four manifolds by gluing the instantons from four spheres to a given four dimensional manifold. This eventually leads to the Donaldson theory, which is the major tool in four manifold theory.

In 1992, Taubes was able to perform similar procedure to construct anti-self dual metrics on any four dimensional manifold as long as we glue in enough copies of $\mathbb{C}P^2$. The twistor space of these manifolds are complex three dimensional manifold fibered over the four manifold with S^2 fibers.

Similar technique was later used by Joyce in 1996 to construct seven dimensional manifolds with holonomy group equal to G_2 and eight dimensional manifolds with holonomy group equal to $Spin(7)$.

The works are all based on singular perturbation method and are very powerful.

Unfortunately the perturbation method is not powerful enough to provide the information of the full moduli space of the corresponding structures. And this is the most basic question in order to apply G_2 manifolds to M-theory.

Existence and moduli space of affine, projective flat and conformally fiat structure in higher dimension is much more difficult than two dimension, for example, it is not known which three manifolds admit affine structures.

A well-known question whether compact affine manifolds have zero Euler number is not solved. It is known to be true if the connection is complete.

Many of the questions are related to developing map from the universal cover of the manifold to $\mathbb{R}^n$, $\mathbb{R}P^n$ or S^n. In general, the map need not be injective. If the map is injective, the manifold with such geometric structure will be equivalent to study of discrete group acting on a domain. Only in one case, we know the developing map is injective. Schoen-Yau proved in 1986, that any conformal map from a complete conformal flat manifold with positive scalar curvature into S^n is injective.

This property is false without assuming positivity of scalar curvature. Conformally flat manifolds with positive scalar curvature are then quotients of domains in S^n by a discrete group of Mobius transformations. The domain is dense in S^n with large codimension.

When symmetry is imposed, we have much better understanding of the spacetime. In the past twenty years, the most fruitful results have been found for spacetime with supersymmetries. The concept of supersymmetry may not be acceptable to some physicists, but it does provide a beautiful and elegant playing ground for geometers. Many classical questions in geometry were resolved by supersymmetric considerations.

A good example is the Seiberg-Witten theory which was motivated by supersymmetric Yang-Mills theory.

The invariant created by Seiberg-Witten theory has been very powerful for the study of four manifolds: especially for those four dimensional symplectic four manifolds.

In the later case, Taubes proved the deep theorem that creates existence of pseudo-holomorphic curves based on the topological data of Seiberg-Witten invariants. As a corollary, he proved that there is only one symplectic structure on $\mathbb{C}P^2$.

A.K. Liu was also able to classify all four dimensional symplectic manifolds that support a metric with positive scalar curvature.