From Boltzmann Equations to Gas Dynamics:

From DiPerna-Lions to Leray

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FROM BOLTZMANN EQUATIONS TO GAS DYNAMICS

I. **INTRODUCTION TO BOLTZMANN EQUATIONS**

II. **GAS DYNAMICAL APPROXIMATIONS**

III. **FROM DIPERNA-LIONS TO LERAY**

1. Boltzmann/DiPerna-Lions
2. Navier-Stokes/Leray
3. Main Results
4. Outline of a Proof
5. Open Problems

1. Boltzmann/DiPerna-Lions

We consider kinetic densities $F(v,x,t)$ over velocities $v \in \mathbb{R}^D$ and positions $x \in \mathbb{T}^D$ at time t that are governed by the scaled Boltzmann initial-value problem

$$\epsilon\, \partial_t F + v \cdot \nabla_x F = \frac{1}{\epsilon} \mathcal{B}(F,F)\,, \qquad F(v,x,0) = F^{in}(v,x) \geq 0\,,$$

where the collision operator $\mathcal{B}$ is given by

$$\mathcal{B}(F,F) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (F_1' F' - F_1 F)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, \mathrm{d}v_1\,.$$

Recall that the F_1', F', F_1, and F appearing in the above integrand designate $F(\cdot,x,t)$ evaluated at the velocities v_1', v', v_1, and v respectively, where the primed velocities are defined by

$$v_1' = v_1 - \omega\, \omega \cdot (v_1 - v)\,, \qquad v' = v + \omega\, \omega \cdot (v_1 - v)\,,$$

for any given $(\omega, v_1, v) \in \mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$.

Collision Kernels

In order to avoid overly technical hypotheses in this presentation, we will restrict ourselves to collision kernels $b(\omega, v_1 - v)$ in the factored form

$$b(\omega, v_1 - v) = \widehat{b}(\omega \cdot n)|v_1 - v|^{\beta}, \tag{1}$$

where $n = (v_1 - v)/|v_1 - v|$. We will require that

$$-D < \beta < 2 \qquad \text{and} \qquad \int_{\mathbb{S}^{D-1}} \widehat{b}(\omega \cdot n)\, \mathrm{d}\omega < \infty\,. \tag{2}$$

This includes the classical collision kernels for elastic hard spheres (for which $\beta = 1$) and for repulsive intermolecular potentials of the form c/r^k with $k > 2\frac{D-1}{D+1}$ (for which $\beta = 1 - 2\frac{D-1}{k}$) that satisfy the weak small-deflection cutoff condition.

In this setting the work of Golse-Saint Raymond (Inventiones 04) treats the case where $\beta = 0$ and $\widehat{b}$ satisfies the more restrictive Grad cutoff condition. This extension of their result is by L-Masmoudi.

We will consider fluid dynamical regimes in which F is close to a spatially homogeneous Maxwellian $M = M(v)$. It can be assumed that this so-called absolute Maxwellian M has the form

$$M(v) \equiv \frac{1}{(2\pi)^{D/2}} \exp(-\tfrac{1}{2}|v|^2) .$$

We define the relative kinetic density $G = G(v, x, t)$ by $F = MG$. The initial-value problem for G is

$$\epsilon \, \partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon} \, \mathcal{Q}(G, G) , \qquad G(v, x, 0) = G^{in}(v, x) , \tag{3}$$

where the collision operator is now given by

$$\begin{aligned} \mathcal{Q}(G, G) &\equiv \frac{1}{M} \mathcal{B}(MG, MG) \\ &= \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (G_1' G' - G_1 G) \, b(\omega, v_1 - v,) \, \mathrm{d}\omega \, M_1 \mathrm{d}v_1 . \end{aligned} \tag{4}$$

Normalizations

The nondimensionalization given last time yields the normalizations

$$\int_{\mathbb{R}^D} M \,\mathrm{d}v = 1 \,, \qquad \int_{\mathbb{T}^D} \mathrm{d}x = 1 \,,$$

associated with the domains $\mathbb{R}^D$ and $\mathbb{T}^D$, the normalization

$$\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v = 1 \,.$$

associated with the collision kernel b, and the normalizations

$$\iint_{\mathbb{R}^D\times\mathbb{T}^D} G^{in}\, M \mathrm{d}v\,\mathrm{d}x = 1 \,, \qquad \iint_{\mathbb{R}^D\times\mathbb{T}^D} v\, G^{in}\, M \mathrm{d}v\,\mathrm{d}x = 0 \,,$$

$$\iint_{\mathbb{R}^D\times\mathbb{T}^D} \tfrac{1}{2}|v|^2 G^{in}\, M \mathrm{d}v\,\mathrm{d}x = \frac{D}{2} \,.$$

associated with the initial data G^{in}.

Notation

In this lecture $\langle \xi \rangle$ will denote the average over $\mathbb{R}^D$ of any integrable function $\xi = \xi(v)$ with respect to the positive unit measure $M\mathrm{d}v$:

$$\langle \xi \rangle = \int_{\mathbb{R}^D} \xi(v)\, M\mathrm{d}v\,.$$

Because $\mathrm{d}\mu = b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1\mathrm{d}v_1\, M\mathrm{d}v$ is a positive unit measure on $\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D$, we denote by $\langle\!\langle \Xi \rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$:

$$\langle\!\langle \Xi \rangle\!\rangle = \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \Xi(\omega, v_1, v)\,\mathrm{d}\mu\,.$$

The measure $d\mu$ is invariant under the coordinate transformations

$$(\omega, v_1, v) \mapsto (\omega, v, v_1)\,, \qquad (\omega, v_1, v) \mapsto (\omega, v_1', v')\,.$$

These, and compositions of these, are called collisional symmetries.

Conservation Properties

Recall that the collision operator has the following property related to the conservation laws of mass, momentum, and energy.

For every measurable ζ the following are equivalent:

- $\zeta \in \operatorname{span}\{1, v_1, \cdots, v_D, \frac{1}{2}|v|^2\}$;
- $\langle \zeta\, \mathcal{Q}(G,G) \rangle = 0$ for "every" G;
- $\zeta_1' + \zeta' - \zeta_1 - \zeta = 0$ for every (ω, v_1, v).

$\boxed{\textbf{Local Conservation Laws}}$

If G is a classical solution of the scaled Boltzmann equation then G satisfies local conservation laws of mass, momentum, and energy:

$$\epsilon\,\partial_t\langle G\rangle + \nabla_x\cdot\,\langle v\,G\rangle = 0\,,$$

$$\epsilon\,\partial_t\langle v\,G\rangle + \nabla_x\cdot\,\langle v\otimes v\,G\rangle = 0\,,$$

$$\epsilon\,\partial_t\langle \tfrac{1}{2}|v|^2 G\rangle + \nabla_x\cdot\,\langle v\tfrac{1}{2}|v|^2 G\rangle = 0\,.$$

Global Conservation Laws

When these are integrated over space and time while recalling the normalizations associated with G^{in}, they yield the global conservation laws of mass, momentum, and energy:

$$\int_{\mathbb{T}^D} \langle G(t) \rangle \, \mathrm{d}x = \int_{\mathbb{T}^D} \langle G^{in} \rangle \, \mathrm{d}x = 1 \, ,$$

$$\int_{\mathbb{T}^D} \langle v \, G(t) \rangle \, \mathrm{d}x = \int_{\mathbb{T}^D} \langle v \, G^{in} \rangle \, \mathrm{d}x = 0 \, ,$$

$$\int_{\mathbb{T}^D} \langle \tfrac{1}{2} |v|^2 G(t) \rangle \, \mathrm{d}x = \int_{\mathbb{T}^D} \langle \tfrac{1}{2} |v|^2 G^{in} \rangle \, \mathrm{d}x = \frac{D}{2} \, .$$

Dissipation Properties

The collision operator has the following properties related to the dissipation of entropy and equilibrium. First, for "every" G

$$\langle \log(G)\, \mathcal{Q}(G,G) \rangle = -\left\langle\!\!\left\langle \frac{1}{4} \log\left(\frac{G_1' G'}{G_1 G}\right) (G_1' G' - G_1 G) \right\rangle\!\!\right\rangle \leq 0\,.$$

Second, for "every" G the following are equivalent:

- $\langle \log(G)\, \mathcal{Q}(G,G) \rangle = 0$;
- $\mathcal{Q}(G,G) = 0$;
- MG is a local Maxwellian.

$\boxed{\textbf{Local Entropy Dissipation Law}}$

If G is a classical solution of the scaled Boltzmann equation then G satisfies local entropy dissipation law:

$$
\begin{aligned}
\epsilon\,\partial_t\langle(G\log(G)-G+1)\rangle+\nabla_x\cdot\,\langle v\,(G\log(G)-G+1)\rangle
&=\frac{1}{\epsilon}\,\langle\log(G)\,\mathcal{Q}(G,G)\rangle\\
&=-\frac{1}{\epsilon}\left\langle\!\!\left\langle\tfrac{1}{4}\log\left(\frac{G_1'G'}{G_1G}\right)(G_1'G'-G_1G)\right\rangle\!\!\right\rangle\\
&\leq 0\,.
\end{aligned}
$$

Global Entropy Dissipation Law

When this is integrated over space and time, it yields the global entropy equality

$$H(G(t)) + \frac{1}{\epsilon^2} \int_0^t R(G(s))\, \mathrm{d}s = H(G^{in}) \,,$$

where the relative entropy functional H is given by

$$H(G) = \int_{\mathbb{T}^D} \langle (G \log(G) - G + 1) \rangle \, \mathrm{d}x \geq 0 \,,$$

while the entropy dissipation rate functional R is given by

$$R(G) = \int_{\mathbb{T}^D} \left\langle\!\!\left\langle \frac{1}{4} \log \left(\frac{G_1' G'}{G_1 G} \right) (G_1' G' - G_1 G) \right\rangle\!\!\right\rangle \mathrm{d}x \geq 0 \,.$$

DiPerna-Lions Theory

The DiPerna-Lions theory gives the existence of a global weak solution to a class of formally equivalent initial-value problems that are obtained by multiplying the Boltzmann equation by $\Gamma'(G)$, where Γ' is the derivative of an admissible function Γ:

$$\Big(\epsilon\,\partial_t + v\cdot\nabla_x\Big)\Gamma(G) = \frac{1}{\epsilon}\,\Gamma'(G)\mathcal{Q}(G,G)\,,$$
$$G(v,x,0) = G^{in}(v,x) \geq 0\,.$$

This is the so-called *renormalized Boltzmann equation*. A differentiable function $\Gamma : [0,\infty) \to \mathbb{R}$ is called *admissible* if for some constant $C_\Gamma < \infty$ it satisfies

$$\Big|\Gamma'(Z)\Big| \leq \frac{C_\Gamma}{\sqrt{1+Z}} \quad \text{for every } Z \geq 0\,.$$

The solutions lie in $C([0,\infty); w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))$, where the prefix "$w$-" on a space indicates that the space is endowed with its weak topology.

Renormalized Solutions

We say that a nonnegative $G \in C([0,\infty); w\text{-}L^1(M \,\mathrm{d}v\,\mathrm{d}x))$ is a weak solution of the renormalized Boltzmann equation provided that it is initially equal to G^{in}, and that for every $Y \in L^\infty(\mathrm{d}v; C^1(\mathbb{T}^D))$ and every $[t_1, t_2] \subset [0,\infty)$ it satisfies

$$\begin{aligned}\epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_2))\, Y\rangle \,\mathrm{d}x - \epsilon \int_{\mathbb{T}^D} &\langle \Gamma(G(t_1))\, Y\rangle \,\mathrm{d}x \\ &- \int_{t_1}^{t_2}\int_{\mathbb{T}^D} \langle \Gamma(G)\, v\cdot\nabla_x Y\rangle \,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{\epsilon}\int_{t_1}^{t_2}\int_{\mathbb{T}^D} \Big\langle \Gamma'(G)\, \mathcal{Q}(G,G)\, Y\Big\rangle \,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

If G is such a weak solution of for one such Γ with $\Gamma' > 0$, and if G satisfies certain bounds, then it is a weak solution for every admissible Γ. Such solutions are called *renormalized solutions* of the Boltzmann equation.

DiPerna-Lions Theorem

Theorem. 1 (DiPerna-Lions) *Let the collision kernel b satisfy (1-2). Given any initial data G^{in} in the entropy class*

$$E(M\,\mathrm{d}v\,\mathrm{d}x) = \left\{ G^{in} \geq 0 \;:\; H(G^{in}) < \infty \right\},$$

there exists at least one $G \geq 0$ in $C([0,\infty); w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))$ that is a renormalized solution of the Boltzmann equation.

This solution also satisfies a weak form of the local conservation law of mass

$$\epsilon\,\partial_t \langle G \rangle + \nabla_x \cdot \langle v\,G \rangle = 0\,.$$

Moreover, there exists a matrix-valued distribution W such that $W\,\mathrm{d}x$ is nonnegative definite measure and such that G and W satisfy a weak form of the local conservation law of momentum

$$\epsilon\,\partial_t\langle v\,G\rangle + \nabla_x\cdot\langle v\otimes v\,G\rangle + \nabla_x\cdot W = 0\,,$$

and for every $t>0$, the global energy equality

$$\int_{\mathbb{T}^D}\langle\tfrac{1}{2}|v|^2G(t)\rangle\,\mathrm{d}x + \int_{\mathbb{T}^D}\tfrac{1}{2}\,\mathrm{tr}(W(t))\,\mathrm{d}x = \int_{\mathbb{T}^D}\langle\tfrac{1}{2}|v|^2G^{in}\rangle\,\mathrm{d}x\,,$$

and the global entropy inequality

$$H(G(t)) + \int_{\mathbb{T}^D}\tfrac{1}{2}\,\mathrm{tr}(W(t))\,\mathrm{d}x + \frac{1}{\epsilon^2}\int_0^t R(G(s))\,\mathrm{d}s \le H(G^{in})\,.$$

2. Navier-Stokes/Leray

We consider fluctuations in bulk velocity $u(x,t)$ and temperature $\theta(x,t)$ over positions $x \in \mathbb{T}^D$ at time t that satisfy the incompressibility condition

$$\nabla_x \cdot u = 0 \,, \tag{5}$$

and are governed by the Navier-Stokes initial-value problem

$$\begin{aligned} &\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u \,, && u(x,0) = u^{in}(x) \,, \\ &\tfrac{D+2}{2}\Big(\partial_t \theta + u \cdot \nabla_x \theta\Big) = \kappa \Delta_x \theta \,, && \theta(x,0) = \theta^{in}(x) \,. \end{aligned} \tag{6}$$

Here $\nu > 0$ and $\kappa > 0$ are the coefficients of viscosity and thermal conductivity.

Hilbert Spaces for the Leray Theory

For the Navier-Stokes system with mean zero initial data, we set the Leray theory in the following Hilbert spaces of vector- and scalar-valued functions:

$$
\begin{aligned}
\mathbb{H}_v &= \left\{ w \in L^2(\mathrm{d}x;\mathbb{R}^D) \,:\, \nabla_x \cdot w = 0\,, \int w \,\mathrm{d}x = 0 \right\}, \\
\mathbb{H}_s &= \left\{ \chi \in L^2(\mathrm{d}x;\mathbb{R}) \,:\, \int \chi \,\mathrm{d}x = 0 \right\}, \\
\mathbb{V}_v &= \left\{ w \in \mathbb{H}_v \,:\, \int |\nabla_x w|^2 \,\mathrm{d}x < \infty \right\}, \\
\mathbb{V}_s &= \left\{ \chi \in \mathbb{H}_s \,:\, \int |\nabla_x \chi|^2 \,\mathrm{d}x < \infty \right\}.
\end{aligned}
$$

Let $\mathbb{H} = \mathbb{H}_v \oplus \mathbb{H}_s$ and $\mathbb{V} = \mathbb{V}_v \oplus \mathbb{V}_s$.

Weak Solutions

We say that $(u,\theta) \in C([0,\infty); w\text{-}\mathbb{H}) \cap L^2(dt;\mathbb{V})$ is a weak solution of the Navier-Stokes system (5-6) provided that it is initially equal to (u^{in},θ^{in}), and that for every $(w,\chi) \in \mathbb{H} \cap C^1$ and every $[t_1,t_2] \subset [0,\infty)$ it satisfies

$$\begin{aligned}\int w\cdot u(t_2)\,\mathrm{d}x - \int w\cdot u(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\!\!\int \nabla_x w : (u\otimes u)\,\mathrm{d}x\,\mathrm{d}t \\ = -\nu \int_{t_1}^{t_2}\!\!\int \nabla_x w : \nabla_x u\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned}$$

$$\begin{aligned}\int \chi\,\theta(t_2)\,\mathrm{d}x - \int \chi\,\theta(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\!\!\int \nabla_x \chi \cdot (u\,\theta)\,\mathrm{d}x\,\mathrm{d}t \\ = -\tfrac{2}{D+2}\,\kappa \int_{t_1}^{t_2}\!\!\int \nabla_x \chi \cdot \nabla_x \theta\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Leray Theorem

Theorem. 2 (Leray) *Given any initial data* $(u^{in}, \theta^{in}) \in \mathbb{H}$*, there exists at least one* $(u, \theta) \in C([0, \infty); w\text{-}\mathbb{H}) \cap L^2(\mathrm{d}t; \mathbb{V})$ *that is a weak solution of the Navier-Stokes system (5-6). Moreover, for every* $t > 0$*,* (u, θ) *satisfies the dissipation inequality*

$$\int \tfrac{1}{2}|u(t)|^2 + \tfrac{D+2}{4}|\theta(t)|^2 \mathrm{d}x + \int_0^t \int \nu |\nabla_x u|^2 + \kappa |\nabla_x \theta|^2 \mathrm{d}x \, \mathrm{d}s \\ \leq \int \tfrac{1}{2}|u^{in}|^2 + \tfrac{D+2}{4}|\theta^{in}|^2 \mathrm{d}x \, .$$

Remark: If for some initial data $u^{in} \in \mathbb{H}_v$ there is a strong solution of the Navier-Stokes motion equation over some time interval then for every $\theta^{in} \in \mathbb{H}_s$ the initial data (u^{in}, θ^{in}) will have a unique Leray solution of the Navier-Stokes system (5-6) and the associated dissipation inequality will be an equality over that time interval.

3. Main Results

Our goal is to show that the Leray Theorem is a consequence of the DiPerna-Lions Theorem. In the formal connection between the Boltzmann equation and the Navier-Stokes system we saw that the fluctuations of the Boltzmann solutions about M should be scaled as ϵ. We will achieve this by considering families G_ϵ in the entropy class $E(M \, \mathrm{d}v \, \mathrm{d}x)$ that satisfy

$$H(G_\epsilon) \leq C^{in} \epsilon^2 \qquad \text{for some } C^{in} \in \mathbb{R}_+ \, ,$$

We define the associated family of fluctuations g_ϵ by

$$G_\epsilon = 1 + \epsilon \, g_\epsilon \, .$$

We can show the family g_ϵ is relatively compact in w-$L^1((1+|v|^2)M \, \mathrm{d}v \, \mathrm{d}x)$ and that every limit point of g_ϵ is in $L^2(M \, \mathrm{d}v \, \mathrm{d}x)$ with

$$\tfrac{1}{2} \int_{\mathbb{T}^D} \langle g^2 \rangle \, \mathrm{d}x \leq \liminf_{\epsilon \to 0} \frac{1}{\epsilon^2} H(G_\epsilon) \leq C^{in} \, .$$

Weak Limit Theorem

Our central result is the following.

Theorem. 3 (Weak Limit Theorem.) *Let the collision kernel b satisfy (1-2). Let $(u^{in}, \theta^{in}) \in \mathbb{H}$.*

Let G_ϵ^{in} be any family in the entropy class $E(M\,\mathrm{d}v\,\mathrm{d}x)$ such that

$$H(G_\epsilon^{in}) \leq C^{in}\epsilon^2 \qquad \text{for some } C^{in} \in \mathbb{R}_+ \,,$$

and such that the associated family of fluctuations g_ϵ^{in} satisfies

$$\lim_{\epsilon \to 0} \Big(\Pi \langle v\, g_\epsilon^{in} \rangle, \langle (\tfrac{1}{D+2}|v|^2 - 1)\, g_\epsilon^{in} \rangle \Big) = (u^{in}, \theta^{in}) \quad \text{in } \mathcal{D}' \,,$$

where Π is the projection in $\mathcal{D}'(\mathbb{T}^D; \mathbb{R}^D)$ onto divergence-free vector fields.

Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation that have G_ϵ^{in} as initial values.

Then the family g_ϵ of fluctuations associated with G_ϵ is relatively compact in w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1((1+|v|^2)M\,\mathrm{d}v\,\mathrm{d}x))$. Every limit point g of g_ϵ has the infinitesimal Maxwellian form

$$g = v \cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta\,,$$

where $(u,\theta) \in C([0,\infty); w\text{-}\mathbb{H}) \cap L^2(\mathrm{d}t; \mathbb{V})$ is a weak solution with initial data (u^{in}, θ^{in}) of the Navier-Stokes system (5-6) with ν and κ given by the classical formulas that also satisfies the dissipation inequality

$$\int \tfrac{1}{2}|u(t)|^2 + \tfrac{D+2}{4}|\theta(t)|^2 \mathrm{d}x + \int_0^t\!\!\int \nu|\nabla_x u|^2 + \kappa|\nabla_x\theta|^2 \mathrm{d}x\,\mathrm{d}s \le C^{in}\,.$$

Moreover, every subsequence g_{ϵ_k} of g_ϵ that converges to g also satisfies the density limits

$$\lim_{k\to\infty}\left(\Pi\langle v\, g_{\epsilon_k}\rangle, \langle(\tfrac{1}{D+2}|v|^2 - 1)\, g_{\epsilon_k}\rangle\right) = (u,\theta) \quad \text{in } C([0,\infty); \mathcal{D}')\,.$$

Remark: Whenever there is a unique Leray solution for the initial data (u^{in}, θ^{in}) over a given time interval, the family g_ϵ will be convergent in w-$L^1(\mathrm{d}t; w\text{-}L^1((1+|v|^2)M\,\mathrm{d}v\,\mathrm{d}x))$ over that time interval.

The weak solutions asserted above may not be Leray solutions because we have not established the correct dissipation inequality. To do so, we need to prepare the initial data better than was done in the previous theorem. The main tool for doing this is the following.

Entropic Convergence

Let G_ϵ be a family in the entropy class $E(M\,\mathrm{d}v\,\mathrm{d}x)$ and let g_ϵ be the associated family of fluctuations. The family g_ϵ is said to *converge entropically at order* ϵ to some $g \in L^2(M\,\mathrm{d}v\,\mathrm{d}x)$ if and only if

$$g_\epsilon \rightarrow g \text{ in } w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x)\,,$$

and

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} H(G_\epsilon) = \int_{\mathbb{T}^D} \tfrac{1}{2}\langle g^2 \rangle \,\mathrm{d}x\,.$$

Remark: One can show that entropic convergence is stronger than norm convergence in $L^1((1+|v|^2)M\,\mathrm{d}v\,\mathrm{d}x)$.

Remark: For every $g \in L^2(M\,\mathrm{d}v\,\mathrm{d}x)$ there exists a family G_ϵ in the entropy class $E(M\,\mathrm{d}v\,\mathrm{d}x)$ such that the associated family of fluctuations g_ϵ converges entropically at order ϵ to g.

Leray Limit Theorem

An important consequence of the Weak Limit Theorem is the following.

Theorem. 4 (Leray Limit Theorem) *Let the collision kernel b satisfy (1-2). Let $(u^{in}, \theta^{in}) \in \mathbb{H}$. and let g^{in} be the infinitesimal Maxwellian given by*

$$g^{in} = v \cdot u^{in} + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta^{in} .$$

Let G_ϵ^{in} be any family in the entropy class $E(M\,\mathrm{d}v\,\mathrm{d}x)$ such that the associated family of fluctuations g_ϵ^{in} satisfies

$$g_\epsilon^{in} \to g^{in} \quad \text{entropically at order } \epsilon \text{ as } \epsilon \to 0 .$$

Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation that have G_ϵ^{in} as initial values.

Then the family g_ϵ of fluctuations associated with G_ϵ is relatively compact in w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1((1+|v|^2)M\,\mathrm{d}v\,\mathrm{d}x))$. Every limit point g of g_ϵ has the infinitesimal Maxwellian form

$$g = v \cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta\,,$$

where $(u,\theta) \in C([0,\infty); w\text{-}\mathbb{H}) \cap L^2(\mathrm{d}t; \mathbb{V})$ is a Leray solution with initial data (u^{in},θ^{in}) of the Navier-Stokes system (5-6) with ν and κ given by the classical formulas.

Remark: Whenever there is a strong Navier-Stokes solution for the initial data (u^{in},θ^{in}) over some time interval, the family $g_\epsilon(t)$ converges entropically at order ϵ as $\epsilon \to 0$ for every t in that time interval.

4. Outline of a Proof

The proof of the Weak Limit Theorem has nine steps:

1) show the limiting fluctuations are infinitesimal Maxwellians,
2) prove the incompressibility and weak Boussinesq conditions,
3) prove the dissipation inequality,
4) obtain a nonlinear compactness by velocity averaging,
5) show that the conservation defects vanish,
6) prove the strong Boussinesq relation,
7) establish compactness of the dynamical fluxes,
8) pass to the limit in the dynamical densities,
9) pass to the limit in the (quadratic) dynamical fluxes.

Limiting Infinitesimal Maxwellians

The fact $H(G_\epsilon(t)) \leq C^{in}\epsilon^2$ implies the family g_ϵ is relatively compact in w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1((1+|v|^2)M\,\mathrm{d}v\,\mathrm{d}x))$.

Let $N_\epsilon = 1 + \epsilon^2 g_\epsilon{}^2$. The fact $\int_0^\infty R(G_\epsilon(s))\,\mathrm{d}s \leq C^{in}\epsilon^4$ implies the family

$$\frac{q_\epsilon}{\sqrt[4]{N_\epsilon}} = \frac{G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon}{\epsilon^2 \sqrt[4]{N_\epsilon}}$$

is relatively compact in w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1((1+|v|^2)\mathrm{d}\mu\,\mathrm{d}x))$. Then from the algebraic identity

$$\frac{g'_{\epsilon 1} + g'_\epsilon - g_{\epsilon 1} - g_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon} = \frac{1}{\epsilon}\,\frac{G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon} - \epsilon\,\frac{g'_{\epsilon 1}g'_\epsilon - g_{\epsilon 1}g_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}\,,$$

we conclude that every limit point g of g_ϵ satisfies $g'_1 + g' - g_1 - g = 0$, whereby it is an infinitesimal Maxwellian.

Approximate Local Conservation Laws

We have to pass to the limit in approximate local conservation laws built from the renormalized Boltzmann equation. We choose to use the normalization of that equation given by

$$\Gamma(Z) = \frac{Z-1}{1+(Z-1)^2}\,.$$

After dividing by ϵ, the renormalized Boltzmann equation becomes

$$\epsilon\,\partial_t \tilde{g}_\epsilon + v\cdot\nabla_x \tilde{g}_\epsilon = \frac{1}{\epsilon^2}\,\Gamma'(G_\epsilon)\,\mathcal{Q}(G_\epsilon, G_\epsilon)\,,$$

where

$$\tilde{g}_\epsilon = \frac{g_\epsilon}{N_\epsilon}\,, \qquad \Gamma'(G_\epsilon) = \frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}\,.$$

One can show that the family $\tilde{g}_\epsilon$ lies in $C([0,\infty); w\text{-}L^2(M\,\mathrm{d}v\,\mathrm{d}x))$.

When the moment of this renormalized Boltzmann equation is taken with respect to any $\zeta \in \operatorname{span}\{1, v_1, \cdots, v_D, |v|^2\}$, one obtains

$$\partial_t \langle \zeta \, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle v \, \zeta \, \tilde{g}_\epsilon \rangle = \frac{1}{\epsilon} \left\langle\!\!\left\langle \zeta \, \Gamma'(G_\epsilon) \, q_\epsilon \right\rangle\!\!\right\rangle .$$

This fails to be a local conservation law because the so-called *conservation defect* on the right-hand side is generally nonzero.

Every DiPerna-Lions solution satisfies this equation in the sense that for every $\chi \in C^1(\mathbb{T}^D)$ and every $[t_1, t_2] \subset [0, \infty)$ one has

$$\begin{aligned} \int \chi \, \langle \zeta \, \tilde{g}_\epsilon(t_2) \rangle \, \mathrm{d}x - \int \chi \, \langle \zeta \, \tilde{g}_\epsilon(t_1) \rangle \, \mathrm{d}x - \int_{t_1}^{t_2} \!\! \int \frac{1}{\epsilon} \nabla_x \chi \cdot \langle v \, \zeta \, \tilde{g}_\epsilon \rangle \, \mathrm{d}x \, \mathrm{d}t & \\ = \int_{t_1}^{t_2} \!\! \int \chi \, \frac{1}{\epsilon} \left\langle\!\!\left\langle \zeta \, \Gamma'(G_\epsilon) \, q_\epsilon \right\rangle\!\!\right\rangle \mathrm{d}x \, \mathrm{d}t \, . & \end{aligned}$$

Approximate global conservation laws are obtained by setting $\chi = 1$.

Incompressibility and Weak Boussinesq

Let $g = \rho + v \cdot u + (\frac{1}{2}|v|^2 - \frac{D}{2})\theta$ be a limit point of g_ϵ. Pass to a sequence g_{ϵ_k} that converges to g and continue to call this sequence as g_ϵ.

If we multiply the approximate local conservation law by ϵ and let $\epsilon \to 0$, we find that for every $\chi \in C^1(\mathbb{T}^D)$ and every $[t_1, t_2] \subset [0, \infty)$ one

$$\int_{t_1}^{t_2} \int \nabla_x \chi \cdot \langle v \, \zeta \, g \rangle \, \mathrm{d}x \, \mathrm{d}t = 0 \,.$$

These are the incompressibility and weak Boussinesq conditions

$$\nabla_x \cdot u = 0 \,, \qquad \qquad \nabla_x (\rho + \theta) = 0 \,.$$

The weak Boussinesq condition implies

$$\rho + \theta = \frac{1}{D} \int_{\mathbb{T}^D} \langle |v|^2 g(t) \rangle \, \mathrm{d}x \,.$$

The right-hand side will vanish once global energy conservation is proved.

Dissipation Inequality

The family q_ϵ/N_ϵ is relatively compact in w-$L^1_{loc}(\mathrm{d}t; w$-$L^1((1+|v|^2)\mathrm{d}\mu\,\mathrm{d}x))$. If q is a limit point of q_ϵ/N_ϵ then q is in $L^2(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t)$ with

$$\int_0^t \int_{\mathbb{T}^D} \tfrac{1}{4}\left\langle\!\!\left\langle q^2 \right\rangle\!\!\right\rangle \mathrm{d}x\,\mathrm{d}s \le \liminf_{\epsilon\to 0} \frac{1}{\epsilon^4}\int_0^t R(G_\epsilon)\,\mathrm{d}s\,.$$

When this is combined with our earlier L^2 bound on g, we obtain

$$\tfrac{1}{2}\int_{\mathbb{T}^D} \langle g(t)^2\rangle\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^D} \tfrac{1}{4}\left\langle\!\!\left\langle q^2 \right\rangle\!\!\right\rangle \mathrm{d}x\,\mathrm{d}s \le \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2} H(G_\epsilon^{in}) \le C^{in}\,.$$

The fact that $g = \rho + v\cdot u + (\frac{1}{2}|v|^2 - \frac{D}{2})\,\theta$ and the fact that

$$v\cdot\nabla_x g = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} q\,b\,\mathrm{d}\omega\,M_1\mathrm{d}v_1$$

can then be used to establish the dissipation inequality after we show that $\rho + \theta = 0$.

Nonlinear Compactness by Averaging

The most difficult part of the proof is showing that

$$\frac{g_\epsilon^2}{\sqrt{N_\epsilon}} \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(aM\,\mathrm{d}v\,\mathrm{d}x))\,,$$

where the attenuation a is given by

$$\begin{aligned} a(v) &= \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1 \\ &= C_\beta \int_{\mathbb{R}^D} |v_1 - v|^\beta\, M_1 \mathrm{d}v_1\,. \end{aligned}$$

One sees that a is a function of $|v|$ only and that $a \sim C_\beta |v|^\beta$ as $|v| \to \infty$.

Remark: This kind of nonlinear compactness was assumed in the early works of Bardos-Golse-L and Lions-Masmoudi. It was first proved for the case $\beta = 0$ by Golse-Saint Raymond (Inventiones 04). The extension given here is by L-Masmoudi.

Let γ_ϵ be defined by $\sqrt{G_\epsilon} = 1 + \epsilon\,\gamma_\epsilon$. Then $g_\epsilon = 2\gamma_\epsilon + \epsilon\,\gamma_\epsilon^{\,2}$ and

$$\frac{g_\epsilon^{\,2}}{1 + \frac{1}{3}\epsilon\,g_\epsilon} = \gamma_\epsilon^{\,2}\frac{4 + 4\epsilon\,\gamma_\epsilon + \epsilon^2\gamma_\epsilon^{\,2}}{1 + \frac{2}{3}\epsilon\,\gamma_\epsilon + \frac{1}{3}\epsilon^2\gamma_\epsilon^{\,2}}\,.$$

Because $\epsilon\,\gamma_\epsilon \geq -1$, it follows from the above that

$$\tfrac{3}{2}\gamma_\epsilon^{\,2} \leq \frac{g_\epsilon^{\,2}}{1 + \frac{1}{3}\epsilon\,g_\epsilon} \leq \tfrac{9}{2}\gamma_\epsilon^{\,2}\,.$$

Because $1/\sqrt{N_\epsilon}$ and $1/(1 + \frac{1}{3}\epsilon\,g_\epsilon)$ are comparable over $\epsilon\,\gamma_\epsilon \geq -1$, the above bound shows that the nonlinear compactness assertion is equivalent to the assertion that

$$a\gamma_\epsilon^{\,2} \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))\,.$$

However, this assertion is a simple consequence once we have established the following two facts. First, that

$$\lim_{R\to\infty} \mathbf{1}_{\{|v|>R\}} a\gamma_\epsilon^2 = 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(M\mathrm{d}v\,\mathrm{d}x)) \text{ uniformly in } \epsilon\,.$$

Second, that for every $R \in \mathbb{R}_+$

$$\mathbf{1}_{\{|v|\le R\}} \gamma_\epsilon^2 \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x))\,.$$

The proof of the first fact rests solely on the entropy dissipation estimate. A key tool is the relative entropy cutoff that was first introduced by Saint-Raymond in her study of the incompressible Euler limit. The proof of the second fact rests on the L^1 velocity averaging lemma of Golse and Saint-Raymond.

Vanishing of the Conservation Defects

The fact that the conservation defect term on the right-hand side above vanishes as $\epsilon \to 0$ follows from the fact χ is bounded, the fact ζ is a collision invariant, and the nonlinear compactness result. Specifically, we can show that

$$\frac{1}{\epsilon} \left\langle\!\!\left\langle \zeta \, \Gamma'(G_\epsilon) \, q_\epsilon \right\rangle\!\!\right\rangle \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)) \text{ as } \epsilon \to 0 \, .$$

In particular, energy is globally conserved. The strong Boussinesq condition thereby holds

$$\rho + \theta = \tfrac{1}{D} \int_{\mathbb{T}^D} \langle |v|^2 g(t) \rangle \, \mathrm{d}x = 0 \, .$$

This also completes the proof of both the infinitesimal Maxwellian form of g and the dissipation inequality. Only establishing the density limits and the Navier-Stokes dynamics remains to be done.

Approximate Dynamical Equations

To obtain the dynamical equations we pass to the limit is the approximate motion and heat equations

$$\partial_t \langle v\, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle A\, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \langle \tfrac{1}{D} |v|^2 \tilde{g}_\epsilon \rangle = \frac{1}{\epsilon} \left\langle\!\!\left\langle v\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\!\right\rangle ,$$

$$\partial_t \langle (\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle B\, \tilde{g}_\epsilon \rangle = \frac{1}{\epsilon} \left\langle\!\!\left\langle (\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\!\right\rangle ,$$

where $A(v) = v \otimes v - \frac{1}{D}|v|^2 I$ and $B(v) = \frac{1}{2}|v|^2 v - \frac{D+2}{2} v$. The difficulty in passing to the limit in these is that the fluxes are order $1/\epsilon$. The approximate motion equation will be integrated against divergence-free test functions. The last term in its flux will thereby be eliminated, and we only have to pass to the limit in the flux terms above that involve A and B — namely, in the terms

$$\frac{1}{\epsilon} \langle A\, \tilde{g}_\epsilon \rangle , \qquad\qquad \frac{1}{\epsilon} \langle B\, \tilde{g}_\epsilon \rangle .$$

Linearized Collision Operator

Now consider the linearized collision operator $\mathcal{L}$ defined by

$$\begin{aligned}\mathcal{L}\tilde{g} &= -\frac{2}{M}\mathcal{B}(M, M\tilde{g}) = -2\mathcal{Q}(1, \tilde{g}) \\ &= \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (\tilde{g} + \tilde{g}_1 - \tilde{g}' - \tilde{g}_1')\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1 \,.\end{aligned}$$

One can show that

$$\frac{1}{a}\mathcal{L} : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v) \quad \text{is self-adjoint and nonnegative definite}\,,$$

and that for every $p \in (1, \infty)$

$$\frac{1}{a}\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \text{is Fredholm}$$

$$\text{with} \quad \mathrm{Null}\left(\tfrac{1}{a}\mathcal{L}\right) = \mathrm{span}\{1, v_1, \cdots, v_D, |v|^2\}\,.$$

New Bilinear Estimates

Let $\Xi = \Xi(\omega, v_1, v)$ be in $L^p(\mathrm{d}\mu)$ for some sufficiently high $p \in [2, \infty)$ ($p = 2$ for $0 \leq \beta < 2$ and $p > 2 - \frac{\beta}{D+\beta}$ for $-D < \beta < 0$). Let $\tilde{g}$ and $\tilde{h}$ be in $L^2(aM\,\mathrm{d}v)$. Then $\Xi\,\tilde{g}_1\,\tilde{h}$ is in $L^1(\mathrm{d}\mu)$ and satisfies the L^1 bound

$$\left\langle\!\!\left\langle |\Xi\,\tilde{g}_1\,\tilde{h}| \right\rangle\!\!\right\rangle \leq C_b^{\frac{1}{p^*}} \left\langle\!\!\left\langle |\Xi|^p \right\rangle\!\!\right\rangle^{\frac{1}{p}} \langle a\,\tilde{g}^2 \rangle^{\frac{1}{2}} \, \langle a\,\tilde{h}^2 \rangle^{\frac{1}{2}} \,,$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\tilde{g}_1$ denotes $\tilde{g}(v_1)$.

Let $\tilde{g}_\epsilon = \tilde{g}_\epsilon(v, x, t)$ and $\tilde{h}_\epsilon = \tilde{h}_\epsilon(v, x, t)$ be families that are bounded in $L^2_{loc}(\mathrm{d}t; L^2(aM\,\mathrm{d}v\,\mathrm{d}x))$. If the family

$$\langle a\tilde{g}_\epsilon^{\,2} \rangle \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))\,,$$

then the family

$$\Xi\,\tilde{g}_{\epsilon 1}\,\tilde{h}_\epsilon \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x))\,.$$

Compactness of the Flux Terms

Because $\langle \zeta\, A \rangle = 0$ and $\langle \zeta\, B \rangle = 0$ for every $\zeta \in \operatorname{span}\{1, v_1, \cdots, v_D, |v|^2\}$, and because $\frac{1}{a} A$ and $\frac{1}{a} B$ are in $L^p(aM\,\mathrm{d}v)$ for every $p \in (1, \infty)$, there exist $\widehat{A}$ and $\widehat{B}$ in $L^p(aM\,\mathrm{d}v)$ for every $p \in (1, \infty)$ that solve

$$
\begin{aligned}
\mathcal{L}\widehat{A} &= A\,, \qquad & \widehat{A} &\perp \operatorname{span}\{1, v_1, \cdots, v_D, |v|^2\}\,,\\
\mathcal{L}\widehat{B} &= B\,, \qquad & \widehat{B} &\perp \operatorname{span}\{1, v_1, \cdots, v_D, |v|^2\}\,.
\end{aligned}
$$

Let $\widehat{\xi}$ be any entry of either $\widehat{A}$ or $\widehat{B}$ and let $\xi = \mathcal{L}\widehat{\xi}$. Then

$$
\langle \xi\, \tilde{g}_\epsilon \rangle = \langle (\mathcal{L}\widehat{\xi})\, \tilde{g}_\epsilon \rangle = \langle \widehat{\xi}\, \mathcal{L}\tilde{g}_\epsilon \rangle = \left\langle\!\!\left\langle \widehat{\xi} \left(\tilde{g}_\epsilon + \tilde{g}_{\epsilon 1} - \tilde{g}'_\epsilon - \tilde{g}'_{\epsilon 1} \right) \right\rangle\!\!\right\rangle .
$$

Next, introduce the symmetrically normalized collision integrand $\tilde{q}_\epsilon$ by

$$
\tilde{q}_\epsilon = \frac{q_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} = \frac{1}{\epsilon^2} \frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}\,,
$$

and define T_ϵ by

$$\frac{1}{\epsilon}\left(\tilde{g}_\epsilon + \tilde{g}_{\epsilon 1} - \tilde{g}'_\epsilon - \tilde{g}'_{\epsilon 1}\right) = \tilde{g}'_{\epsilon 1}\tilde{g}'_\epsilon - \tilde{g}_{\epsilon 1}\tilde{g}_\epsilon - \tilde{q}_\epsilon + T_\epsilon \,.$$

The flux moments decompose as

$$\frac{1}{\epsilon}\langle \xi\, \tilde{g}_\epsilon\rangle = \left\langle\!\!\left\langle (\widehat{\xi}' - \widehat{\xi})\, \tilde{g}_{\epsilon 1}\tilde{g}_\epsilon \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \widehat{\xi}\, \tilde{q}_\epsilon \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \widehat{\xi}\, T_\epsilon \right\rangle\!\!\right\rangle .$$

The first term in this decomposition is quadratic in $\tilde{g}_\epsilon$, the second is linear in $\tilde{q}_\epsilon$, while the last is a remainder. We control each of these terms separately.

Our nonlinear compactness result and our bilinear result imply that

$$\left\langle\!\!\left\langle \widehat{\xi}\, T_\epsilon \right\rangle\!\!\right\rangle \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)) \text{ as } \epsilon \to 0\,.$$

This controls the last term in our flux moment decomposition.

We then show that as $\epsilon \to 0$ one has

$$\left\langle\!\!\left\langle \widehat{\xi}\, \tilde{q}_\epsilon \right\rangle\!\!\right\rangle \to \langle \widehat{\xi}\, A \rangle : \nabla_x u + \langle \widehat{\xi}\, B \rangle \cdot \nabla_x \theta \quad \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x))\,.$$

This controls the linear term in our decomposition. In particular, we see that as $\epsilon \to 0$ one has

$$\begin{aligned} \left\langle\!\!\left\langle \widehat{A}\, \tilde{q}_\epsilon \right\rangle\!\!\right\rangle &\to \nu \left[\nabla_x u + (\nabla_x u)^T \right] \quad && \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x; \mathbb{R}^{D \vee D}))\,, \\ \left\langle\!\!\left\langle \widehat{B}\, \tilde{q}_\epsilon \right\rangle\!\!\right\rangle &\to \kappa\, \nabla_x \theta && \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x; \mathbb{R}^{D}))\,, \end{aligned}$$

where ν and κ are given by the classical formulas.

Our nonlinear compactness result and our bilinear result imply that

$$\left\langle\!\!\left\langle (\widehat{\xi}' - \widehat{\xi})\, \tilde{g}_{\epsilon 1} \tilde{g}_\epsilon \right\rangle\!\!\right\rangle \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))\,.$$

This controls the quadratic term in our decomposition, whereby the flux moments involving A and B are similarly relatively compact.

Convergence of the Densities

The densities in our approximate motion and heat equations are

$$\Pi\langle v\,\tilde{g}_\epsilon\rangle \quad \text{and} \quad \langle(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\,\tilde{g}_\epsilon\rangle\,,$$

where Π is the projection onto divergence-free vector fields in $L^2(\mathrm{d}x;\mathbb{R}^D)$. These are convergent in $w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x))$.

We use the Arzela-Ascoli Theorem and the just established flux controls to establish that as $\epsilon \to 0$ one has

$$\begin{aligned} \Pi\langle v\,\tilde{g}_\epsilon\rangle &\to u && \text{in } C([0,\infty); w\text{-}L^2(\mathrm{d}x;\mathbb{R}^D))\,, \\ \langle(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\,\tilde{g}_\epsilon\rangle &\to \tfrac{D+2}{2}\theta && \text{in } C([0,\infty); w\text{-}L^2(\mathrm{d}x))\,. \end{aligned}$$

The density limits asserted in the Weak Limit Theorem then follow.

Convergence of the Quadratic Flux Terms

Now, we concentrate on the passage to the limit in the quadratic term in our flux moment decomposition. This term has the equivalent forms

$$\left\langle\!\!\left\langle (\widehat{\xi}' - \widehat{\xi})\, \tilde{g}_{\epsilon 1} \tilde{g}_{\epsilon} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \widehat{\xi} \Big(\tilde{g}'_{\epsilon 1} \tilde{g}'_{\epsilon} - \tilde{g}_{\epsilon 1} \tilde{g}_{\epsilon} \Big) \right\rangle\!\!\right\rangle = \Big\langle \widehat{\xi}\, \mathcal{Q}(\tilde{g}_{\epsilon}, \tilde{g}_{\epsilon}) \Big\rangle \,.$$

We decompose $\tilde{g}_{\epsilon}$ into its infinitesimal Maxwellian $\mathcal{P}\tilde{g}_{\epsilon}$ and its deviation $\mathcal{P}^{\perp}\tilde{g}_{\epsilon}$ as

$$\tilde{g}_{\epsilon} = \mathcal{P}\tilde{g}_{\epsilon} + \mathcal{P}^{\perp}\tilde{g}_{\epsilon} \,,$$

where $\mathcal{P}^{\perp} = I - \mathcal{P}$ and $\mathcal{P}$ is defined by

$$\mathcal{P}\tilde{g} = \langle \tilde{g} \rangle + v \cdot \langle v\, \tilde{g} \rangle + \Big(\tfrac{1}{2}|v|^2 - \tfrac{D}{2} \Big) \langle (\tfrac{1}{D}|v|^2 - 1)\, \tilde{g} \rangle \,.$$

We then use entropy dissipation estimates and the coercivity of $\mathcal{L}$ to show that $\langle a(\mathcal{P}^{\perp}\tilde{g}_{\epsilon})^2 \rangle$ vanishes as $\epsilon \to 0$.

When this decomposition is placed into the quadratic flux term, it yields

$$\begin{aligned}\left\langle \hat{\xi}\, \mathcal{Q}(\tilde{g}_\epsilon, \tilde{g}_\epsilon) \right\rangle = \left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}\tilde{g}_\epsilon) \right\rangle &+ 2 \left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}^\perp \tilde{g}_\epsilon) \right\rangle \\ &+ \left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}^\perp\tilde{g}_\epsilon, \mathcal{P}^\perp \tilde{g}_\epsilon) \right\rangle .\end{aligned}$$

The last two terms above vanish as $\epsilon \to 0$ (by our bilinear bounds) because $\langle a(\mathcal{P}^\perp \tilde{g}_\epsilon)^2 \rangle$ vanishes as $\epsilon \to 0$.

Because $\mathcal{P}\tilde{g}_\epsilon$ is an infinitesimal Maxwellian, we can express the first term above as

$$\begin{aligned}\left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}\tilde{g}_\epsilon) \right\rangle &= \tfrac{1}{2}\left\langle \hat{\xi}\, \mathcal{L}\Big(\mathcal{P}\tilde{g}_\epsilon\Big)^2 \right\rangle = \tfrac{1}{2}\left\langle (\mathcal{P}^\perp \xi) \Big(\mathcal{P}\tilde{g}_\epsilon\Big)^2 \right\rangle \\ &= \tfrac{1}{2}\langle \xi\, A\rangle : (\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon) + \langle \xi\, B\rangle \cdot \tilde{u}_\epsilon \tilde{\theta}_\epsilon + \tfrac{1}{2}\langle \xi\, C\rangle\, \tilde{\theta}_\epsilon^2 \,,\end{aligned}$$

where $C(v) = \frac{1}{4}|v|^4 - \frac{D+2}{2}|v|^2 + \frac{D(D+2)}{4}$ and

$$\tilde{\rho}_\epsilon = \langle \tilde{g}_\epsilon \rangle \,, \qquad \tilde{u}_\epsilon = \langle v\, \tilde{g}_\epsilon \rangle \,, \qquad \tilde{\theta}_\epsilon = \langle (\tfrac{1}{D}|v|^2 - 1)\, \tilde{g}_\epsilon \rangle \,.$$

We thereby have reduced the problem to passing to the limit in the terms

$$\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon \,, \qquad \tilde{u}_\epsilon \tilde{\theta}_\epsilon \,, \qquad \tilde{\theta}_\epsilon^2 \,. \tag{7}$$

We are unable to pass to the limit in the above terms in full generality. Rather, we can show that

$$\left.\begin{aligned} \lim_{\epsilon\to 0} \Pi \nabla_x \cdot \left(\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon \right) &= \Pi \nabla_x \cdot (u \otimes u) \\ \lim_{\epsilon\to 0} \nabla_x \cdot \left(\tilde{\theta}_\epsilon \, \tilde{u}_\epsilon \right) &= \nabla_x \cdot (\theta \, u) \end{aligned}\right\} \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)) \,,$$

where Π is the projection onto divergence-free vector fields in $L^2(\mathrm{d}x; \mathbb{R}^D)$. It follows that

$$\left.\begin{aligned} \lim_{\epsilon\to 0} \Pi \nabla_x \cdot \left\langle \widehat{A} \, \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}\tilde{g}_\epsilon) \right\rangle &= \Pi \nabla_x \cdot (u \otimes u) \\ \lim_{\epsilon\to 0} \nabla_x \cdot \left\langle \widehat{B} \, \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}\tilde{g}_\epsilon) \right\rangle &= \tfrac{D+2}{2} \nabla_x \cdot (\theta \, u) \end{aligned}\right\} \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)) \,.$$

We thereby obtain the limiting fluxes for the Navier-Stokes-Fourier motion and heat equations, thereby completing our proof.

5. Open Problems

- Establishing the limit for

$$\frac{1}{\epsilon}\mathcal{P}^{\perp} g_\epsilon \,.$$

  This would require figuring out how to pass to the limit in $\tilde{\theta}_\epsilon^2$ in the above proof, but perhaps there is another way to do it.

- Establishing the local Leray dissipation inequality in the limit.

- Looking for additional regularity for Leary solutions in the limit.

- Don't forget the open problems mentioned last time!

Thank you!