From Boltzmann Equations to Gas Dynamics:

Gas Dynamical Approximations

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7 March 2007
Gill Distinguished Lecture Series
University of California, Los Angeles
Los Angeles, California, 5,7,9 March 2007

FROM BOLTZMANN EQUATIONS TO GAS DYNAMICS

1. Nondimensionalization

The Knudsen number is a nondimensional parameter that quantifies when a gas is in a fluid dynamical regime. In 1912 Hilbert proposed that it should be at the heart of every systematic derivation of any fluid dynamical system. Consider the Boltzmann initial-value problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F)\,, \qquad F(v, x, 0) = F^{in}(v, x)\,.$$

In order to remove complications due to boundaries, we consider the spatial domain Ω to be a periodic box $\mathbb{T}^D$. The dimensional scales of this problem can be identified as follows.

First, the volume of the periodic box determines a length scale λ_o by setting

$$\int \mathrm{d}x = \lambda_o^D\,.$$

The sides of the box Ω need not all have the same length; however, all of these lengths are assumed to be of the same order.

Next, we determine two other dimensional scales from the initial data F^{in}. One can always find a Galilean frame in which

$$\iint v\, F^{in}\, \mathrm{d}v\, \mathrm{d}x = 0\,.$$

Therefore one cannot obtain a meaningful dimensional scale from the total momentum of the initial data. However, the total mass and total energy of the initial data F^{in} determine a density scale ρ_o and a velocity scale $\theta_o^{1/2}$ by the relations

$$\iint F^{in}\, \mathrm{d}v\, \mathrm{d}x = \rho_o \lambda_o^D\,, \qquad \iint \tfrac{1}{2}|v|^2 F^{in}\, \mathrm{d}v\, \mathrm{d}x = \tfrac{D}{2}\rho_o \theta_o \lambda_o^D\,.$$

The equilibrium associated with the initial data F^{in} is given by

$$M_o \equiv \mathcal{M}(\rho_o, 0, \theta_o) = \frac{\rho_o}{(2\pi\theta_o)^{D/2}} \exp\left(-\frac{|v|^2}{2\theta_o} \right).$$

Finally, the collision operator determines a time scale τ_o by

$$\iiint M_{o1} M_o \, b(\omega, v_1 - v) \, \mathrm{d}\omega \, \mathrm{d}v_1 \, \mathrm{d}v = \frac{\rho_o}{\tau_o} \, .$$

This time is on the order of the time interval that molecules in the equilibrium density M_o spend traveling freely between collisions, the so-called mean free time. The time scale τ_o times the velocity scale $\theta_o^{1/2}$ therefore gives the length scale of the mean free path.

Now that the dimensional scales of the Boltzmann initial-value problem have been identified, it can be reformulated in terms of nondimensional variables. These are introduced below adorned with hats. Nondimensional velocity, space, and time are defined by

$$v = \theta_o^{1/2} \, \widehat{v} \, , \qquad x = \lambda_o \, \widehat{x} \, , \qquad t = \frac{\lambda_o}{\theta_o^{1/2}} \, \widehat{t} \, . \tag{1}$$

A nondimensional kinetic density is then given by

$$F(v, x, t) = \frac{\rho_o}{\theta_o^{D/2}} \widehat{F}(\hat{v}, \hat{x}, \hat{t}\,)\,,$$

while a nondimensional collision operator is given by

$$\mathcal{B}(F, F)(v, x, t) = \frac{\rho_o}{\tau_o \theta_o^{D/2}} \widehat{\mathcal{B}}(\widehat{F}, \widehat{F})(\hat{v}, \hat{x}, \hat{t}\,)\,.$$

This nondimensional collision operator has the form

$$\widehat{\mathcal{B}}(\hat{f}, \hat{f}) = \iint \left(\hat{f}_1' \hat{f}' - \hat{f}_1 \hat{f}\right) \hat{b}\, \mathrm{d}\omega\, d\hat{v}_1\,,$$

where the nondimensional collision kernel is given by

$$b(\omega, v_1 - v) = \frac{1}{\rho_o \tau_o} \hat{b}(\omega, \hat{v}_1 - \hat{v})\,.$$

Substituting these relations into the Boltzmann equation and henceforth dropping all hats yields the nondimensional initial-value problem

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\epsilon} \mathcal{B}(F, F) \,, \qquad F(v, x, t) = F^{in}(v, x) \,.$$

where $\epsilon = \theta_o^{1/2} \tau_o / \lambda_o$ is the Knudsen number. The Knudsen number is the ratio of the scale of mean-free-paths to the macroscopic length scale. Fluid dynamical regimes are characterized by the Knudsen number being small. We consider families of solutions F_ϵ parametrized by ϵ.

The nondimensional equilibrium associated with this problem is the so-called unit Maxwellian

$$M \equiv \mathcal{M}(1, 0, 1) = \frac{1}{(2\pi)^{D/2}} \exp\left(- \frac{|v|^2}{2} \right) ,$$

2. Linear and Weakly Nonlinear Systems

We consider fluid dynamical regimes in which F is close to the spatially homogeneous Maxwellian $M = M(v)$. By an appropriate choice of a Galilean frame and of mass and velocity units, it can be assumed that this Maxwellian has the form

$$M(v) \equiv \mathcal{M}(v; 1, 0, 1) = \frac{1}{(2\pi)^{D/2}} \exp(-\tfrac{1}{2}|v|^2) \,.$$

This corresponds to the spatially homogeneous fluid state with mass density and temperature equal to 1 and bulk velocity equal to 0.

We then seek solutions of the form

$$F_\epsilon(v, x, t) = M(v)\Big(1 + \delta_\epsilon \, g_\epsilon(v, x, t)\Big) \,,$$

where δ_ϵ satisfies $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

Infinitesimal Maxwellians

Assuming that g_ϵ converges to g as $\epsilon \to 0$, one can show that

$$g_\epsilon \to g = \rho + v \cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\theta \,,$$

where $\rho(x,t)$, $u(x,t)$, and $\theta(x,t)$ are fluctuations of the mass density, bulk velocity, and temperature about their equilibrium values: 1, 0, and 1. This limiting form is a so-called *infinitesimal Maxwellian* because

$$\mathcal{M}(v; 1 + \delta_\epsilon \rho, \delta_\epsilon u, 1 + \delta_\epsilon \theta) = M(v)\Big(1 + \delta_\epsilon g + O(\delta_\epsilon{}^2)\Big) \,.$$

It is the limiting form shared by the leading orders of all linear and weakly nonlinear fluid dynamical approximations.

Acoustic Scaling

Upon passing to the limit in the local conservation laws associated with

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} \mathcal{B}(F_\epsilon, F_\epsilon)\,,$$

one finds that the fluctuations ρ, u, and θ satisfy

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot u &= 0\,, \\ \partial_t u + \nabla_x (\rho + \theta) &= 0\,, \\ \tfrac{D}{2} \partial_t \theta + \nabla_x \cdot u &= 0\,. \end{aligned}$$

This is the acoustic system. It is the linearization about the homogeneous state of the compressible Euler system. It is one of the simplest systems of fluid dynamical equations imaginable, being essentially the wave equation.

Derivation of Incompressible Systems

It is easily seen that when ρ, u, and θ satisfy

$$\nabla_x \cdot u = 0\,, \qquad \nabla_x(\rho + \theta) = 0\,,$$

they are a stationary solution of the acoustic system which will generally vary in space. On the other hand, it can be shown that absolute Maxwellians are the only stationary solutions of the Boltzmann equation.

It is clear that the time scale at which the acoustic system was derived was not long enough to see the evolution of these solutions. We therefore consider the Boltzmann equation over a longer time scale $1/\tau_\epsilon$:

$$\tau_\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\,\mathcal{B}(F_\epsilon, F_\epsilon)\,,$$

where $\tau_\epsilon \to 0$ as $\epsilon \to 0$.

One can give formal moment derivations of three incompressible fluid dynamical systems, depending on the limiting behavior of the ratio δ_ϵ/ϵ as $\epsilon \longrightarrow 0$:

- When $\delta_\epsilon/\epsilon \longrightarrow 0$, one considers time scales on the order of $1/\epsilon$ (by choosing $\tau_\epsilon = \epsilon$) and an incompressible Stokes system is derived.

- When $\delta_\epsilon/\epsilon \longrightarrow 1$ (or any other nonzero number), one considers time scales on the order of $1/\epsilon$ ($\tau_\epsilon = \epsilon$) and an incompressible Navier-Stokes system is derived.

- When $\delta_\epsilon/\epsilon \longrightarrow \infty$, one considers time scales on the order of $1/\delta_\epsilon$ ($\tau_\epsilon = \delta_\epsilon$), and an incompressible Euler system is derived.

What underlies this result is the fact that δ_ϵ/ϵ is the Reynolds number.

Incompressible Stokes System

$$\nabla_x \cdot u = 0 \,, \qquad \rho + \theta = 0 \,.$$

$$\partial_t u + \nabla_x p = \mu \Delta_x u \,, \qquad u(x,0) = u^{in}(x) \,,$$
$$\tfrac{D+2}{2} \partial_t \theta = \kappa \Delta_x \theta \,, \qquad \theta(x,0) = \theta^{in}(x) \,.$$

Here $\mu > 0$ and $\kappa > 0$ are the viscosity and thermal conductivity coefficients. Like the acoustic system, the Stokes system is also one of the simplest systems of fluid dynamical equations imaginable, being essentially a system of linear heat equations.

Incompressible Navier-Stokes System

$$\nabla_x \cdot u = 0 \,, \qquad \rho + \theta = 0 \,.$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \mu \Delta_x u \,, \qquad u(x,0) = u^{in}(x) \,,$$
$$\tfrac{D+2}{2}\Big(\partial_t \theta + u \cdot \nabla_x \theta\Big) = \kappa \Delta_x \theta \,, \qquad \theta(x,0) = \theta^{in}(x) \,.$$

Here the viscosity and thermal conductivity coefficients, μ and κ, have the same values as in the Stokes system. Unlike the Stokes system however, the Navier-Stokes system is nonlinear. While this fact does not complicate its formal derivation, it makes the mathematical establishment of its validity much harder.

Incompressible Euler System

$$\nabla_x \cdot u = 0\,, \qquad \rho + \theta = 0\,.$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = 0\,, \qquad u(x,0) = u^{in}(x)\,,$$
$$\tfrac{D+2}{2}\Big(\partial_t \theta + u \cdot \nabla_x \theta\Big) = 0\,, \qquad \theta(x,0) = \theta^{in}(x)\,.$$

Like the Navier-Stokes system, the Euler system is nonlinear. The full mathematical establishment of its validity is also an open problem.

As was the case for the acoustic system, the Euler system has stationary solutions that vary in space. It is clear that the time scale at which the Euler system was derived was not long enough to see the evolution of these solutions. Even at a formal level it is unclear how this long-time evolution should be governed.

Weakly Compressible Systems

A fluid dynamical system that formally includes both the acoustic and the Stokes limits is the so-called *weakly compressible Stokes system*

$$
\begin{aligned}
\partial_t \rho_\epsilon + \nabla_x \cdot u_\epsilon &= 0\,, \\
\partial_t u_\epsilon + \nabla_x (\rho_\epsilon + \theta_\epsilon) &= \epsilon \mu \nabla_x \cdot \left[\nabla_x u_\epsilon + (\nabla_x u_\epsilon)^T - \tfrac{2}{D} \nabla_x \cdot u_\epsilon I \right], \\
\tfrac{D}{2} \partial_t \theta_\epsilon + \nabla_x \cdot u_\epsilon &= \epsilon \kappa \Delta_x \theta_\epsilon \,.
\end{aligned}
$$

Notice that ϵ appears in the approximating system.

A so-called *weakly compressible Navier-Stokes system* that formally includes both the acoustic and the Navier-Stokes limits is much harder to write down. Jiang-L have derived such a system from the Boltzmann equation. It decomposes into a component governed by the incompressible Navier-Stokes system and a component governed by a nonlocal quadratic acoustic equation that couples to the incompressible component.

$\boxed{\textbf{Remarks}}$

Finally, it should be pointed out that the above systems are not the only incompressible Stokes, Navier-Stokes, and Euler systems that may be derived as fluid dynamical limits of the Boltzmann equation. More refined asymptotic balances lead to incompressible Stokes, Navier-Stokes, and Euler systems that differ from those above in that (1) the heat equation includes a viscous heating term and (2) the Boussinesq relation is replaced by $p = \rho + \theta$. These also have moment-based derivations. One should therefore be careful about referring to "the incompressible Stokes system" (for example) until it is clear to which Stokes system you are referring.

In addition, there are other systems that have moment-based derivations, including some that are "beyond Navier-Stokes". The goals of the BGL program (1989) are (1) to identify those fluid dynamical systems that can be so derived, and (2) to give a full mathematical justification of those formal derivations.

3. Global Solutions

We now make more precise: (1) the notion of solution for the Boltzmann equation, and (2) the notion of solution for the fluid dynamical systems. Ideally, these solutions should be global while the bounds should be physically natural.

We therefore work in the setting of DiPerna-Lions renormalized solutions for the Boltzmann equation, and in the setting of Leray solutions for the Navier-Stokes system. These theories have the virtues of considering physically natural classes of initial data, and consequently, of yielding global solutions.

One of the main goals of the BGL program was to connect the DiPerna-Lions theory of renormalized solutions of the Boltzmann equation to the Leray theory of weak solutions of the incompressible Navier-Stokes system.

DiPerna-Lions Theory

The DiPerna-Lions theory gives the existence of a global weak solution to a class of formally equivalent initial-value problems that are obtained by multiplying the Boltzmann equation by $\Gamma'(G)$, where Γ' is the derivative of an admissible function Γ:

$$\Big(\tau_\epsilon\,\partial_t + v\cdot\nabla_x\Big)\Gamma(F) = \frac{1}{\epsilon}\,\Gamma'(F)\mathcal{B}(F,F)\,,$$
$$F(v,x,0) = F^{in}(v,x) \geq 0\,.$$

This is the so-called renormalized Boltzmann equation. A differentiable function $\Gamma : [0,\infty) \to \mathbb{R}$ is called *admissible* if for some constant $C_\Gamma < \infty$ it satisfies

$$\Big|\Gamma'(Z)\Big| \leq \frac{C_\Gamma}{\sqrt{1+Z}} \quad \text{for every } Z \geq 0\,.$$

The solutions lie in $C([0,\infty); w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))$, where the prefix "$w$-" on a space indicates that the space is endowed with its weak topology.

DiPerna-Lions Theorem - 1

Theorem. 1 (DiPerna-Lions Renormalized Solutions) *Let b satisfy*

$$\lim_{|v|\to\infty} \frac{1}{1+|v|^2} \int_{\mathbb{S}^{D-1}\times K} b(\omega, v_1 - v)\,\mathrm{d}\omega\,\mathrm{d}v_1 = 0$$

for every compact $K \subset \mathbb{R}^D$.

Given any initial data F^{in} in the entropy class

$$E(M\,\mathrm{d}v\,\mathrm{d}x) = \left\{ F^{in} \geq 0 \,:\, H(F^{in}) < \infty \right\},$$

there exists at least one $F \geq 0$ in $C([0,\infty); w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))$ that for every admissible function Γ is a weak solution of renormalized Boltzmann equation.

This solution satisfies a weak form of the local conservation law of mass

$$\tau_\epsilon \, \partial_t \langle F \rangle + \nabla_x \cdot \langle v \, F \rangle = 0 \, .$$

Moreover, there exists a matrix-valued distribution W *such that* $W \, \mathrm{d}x$ *is a nonnegative definite measure and* G *and* W *satisfy a weak form of the local conservation law of momentum*

$$\tau_\epsilon \, \partial_t \langle v \, F \rangle + \nabla_x \cdot \langle v \otimes v \, F \rangle + \nabla_x \cdot W = 0 \, ,$$

and for every $t > 0$*, the global energy equality*

$$\int \langle \tfrac{1}{2} |v|^2 F(t) \rangle \, \mathrm{d}x + \int \tfrac{1}{2} \, \mathrm{tr}(W(t)) \, \mathrm{d}x = \int \langle \tfrac{1}{2} |v|^2 F^{in} \rangle \, \mathrm{d}x \, ,$$

and the global entropy inequality

$$H(F(t)) + \int \tfrac{1}{2}\,\mathrm{tr}(W(t))\,\mathrm{d}x + \frac{1}{\tau_\epsilon \epsilon} \int_0^t R(F(s))\,\mathrm{d}s \leq H(F^{in})\,,$$

where the relative entropy functional H *is given by*

$$H(F) = \int \left\langle F \log\left(\frac{F}{M}\right) - F + M \right\rangle \mathrm{d}x \geq 0\,,$$

while the entropy dissipation rate functional R *is given by*

$$R(F) = \iiiint \tfrac{1}{4} \log\left(\frac{F_1' F'}{F_1 F}\right) (F_1' F' - F_1 F)\, b\,\mathrm{d}\omega\,\mathrm{d}v_1\,\mathrm{d}v\,\mathrm{d}x \geq 0\,.$$

DiPerna-Lions Theorem - 3

Remarks: DiPerna-Lions renormalized solutions are very weak — much weaker than standard weak solutions. They are not known to satisfy many properties that one would formally expect to be satisfied by solutions of the Boltzmann equation. In particular, the theory does not assert either

- the formally expected local momentum conservation law,
- the formally expected global energy conservation law,
- any local energy conservation law or inequality,
- any global entropy equality,
- any local entropy inequality,
- or the uniqueness of the solution.

Leray Theory

The DiPerna-Lions theory has many similarities with the Leray theory of global weak solutions of the initial-value problem for Navier-Stokes type systems. For the Navier-Stokes system with mean zero initial data, we set the Leray theory in the following Hilbert spaces of vector- and scalar-valued functions:

$$
\begin{aligned}
\mathbb{H}_v &= \left\{ w \in L^2(\mathrm{d}x; \mathbb{R}^D) \,:\, \nabla_x \cdot w = 0 \,, \int w \, \mathrm{d}x = 0 \right\} , \\
\mathbb{H}_s &= \left\{ \chi \in L^2(\mathrm{d}x; \mathbb{R}) \,:\, \int \chi \, \mathrm{d}x = 0 \right\} , \\
\mathbb{V}_v &= \left\{ w \in \mathbb{H}_v \,:\, \int |\nabla_x w|^2 \, \mathrm{d}x < \infty \right\} , \\
\mathbb{V}_s &= \left\{ \chi \in \mathbb{H}_s \,:\, \int |\nabla_x \chi|^2 \, \mathrm{d}x < \infty \right\} .
\end{aligned}
$$

Let $\mathbb{H} = \mathbb{H}_v \oplus \mathbb{H}_s$ and $\mathbb{V} = \mathbb{V}_v \oplus \mathbb{V}_s$.

Leray Theorem

Theorem. 2 (Leray Solutions) *Given any initial data* $(u^{in}, \theta^{in}) \in \mathbb{H}$*, there exists at least one* $(u, \theta) \in C([0,\infty); w\text{-}\mathbb{H}) \cap L^2(dt; \mathbb{V})$ *that is a weak solution of the Navier-Stokes system. Moreover, for every* $t > 0$*,* (u, θ) *satisfies the dissipation inequalities*

$$\int \tfrac{1}{2}|u(t)|^2 \mathrm{d}x + \int_0^t \int \nu |\nabla_x u|^2 \mathrm{d}x\, \mathrm{d}s \le \int \tfrac{1}{2}|u^{in}|^2 dx\,,$$
$$\int \tfrac{D+2}{4}|\theta(t)|^2 \mathrm{d}x + \int_0^t \int \kappa |\nabla_x \theta|^2 \mathrm{d}x\, \mathrm{d}s \le \int \tfrac{D+2}{4}|\theta^{in}|^2 \mathrm{d}x\,.$$

Remarks: By arguing formally from the Navier-Stokes system, one would expect these inequalities to be equalities. However, that is not asserted by the Leray theory. Also, as was the case for the DiPerna-Lions theory, the Leray theory does not assert uniqueness of the solution.

$\boxed{\textbf{A Variant of Leray Theory}}$

Because the role of the above dissipation inequalities is to provide a-priori estimates, the existence theory also works if they are replaced by the single dissipation inequality

$$\begin{aligned}\int \tfrac{1}{2}|u(t)|^2 + \tfrac{D+2}{4}|\theta(t)|^2 dx + \int_0^t\int \nu|\nabla_x u|^2 + \kappa|\nabla_x\theta|^2 dx\, ds \\ \leq \int \tfrac{1}{2}|u^{in}|^2 + \tfrac{D+2}{4}|\theta^{in}|^2 dx\,.\end{aligned}$$

It is this version of the Leray theory that we will obtain in the limit.

Weakly Compressible Navier-Stokes System

Jiang-L have shown that the weakly compressible Navier-Stokes system has a global weak solution in L^2. This result includes the Leray theory, so it cannot be improved easily. As with the Leray theory, the key to this result is an “energy” dissipation estimate. Indeed, this global existence result is very general. Jiang also has used a Littlewood-Payly decomposition to show the acoustic part is unique for a given incompressible component.

The general setting for their existence result is the following.

Let $U \mapsto H(U)$ be a strictly convex entropy for the system

$$\partial_t U + \nabla_x \cdot F(U) = \epsilon \nabla_x \cdot \Big[D(U) \nabla_x H_U(U) \Big] \,.$$

This means that there exist $J(U)$ such that (Friedrichs-Lax)

$$H_U(U) F_U(U) = J_U(U) \,,$$

and that

$$\nabla_x H_U(U) \cdot D(U) \nabla_x H_U(U) \geq 0 \,.$$

Hence, one has the local dissipation law

$$\begin{aligned} \partial_t H(U) + \nabla_x \cdot J(U) = \epsilon \, \nabla_x \cdot \Big[H_U(U) \, D(U) \nabla_x H_U(U) \Big] \\ - \epsilon \, \nabla_x H_U(U) \cdot D(U) \nabla_x H_U(U) \,. \end{aligned}$$

The weakly nonlinear approximation of the solution U_ϵ to this system near a constant solution U_o is $U_\epsilon = U_o + \epsilon \widetilde{U}_\epsilon$ where $\widetilde{U}_\epsilon$ satisfies

$$\partial_t \widetilde{U}_\epsilon + A \cdot \nabla_x \widetilde{U}_\epsilon + \epsilon \nabla_x \cdot \overline{Q}(\widetilde{U}_\epsilon, \widetilde{U}_\epsilon) = \epsilon \nabla_x \cdot \left[\overline{D} \nabla_x \widetilde{U}_\epsilon \right],$$

where $A = F_U(U_o)$,

$$\overline{Q}(V,V) = \lim_{T\to\infty} \frac{1}{2T} \int_0^T e^{tA\,\cdot\,\nabla_x} F_{UU}(U_o)(e^{-tA\,\cdot\,\nabla_x} V, e^{-tA\,\cdot\,\nabla_x} V)\,\mathrm{d}t$$

$$\overline{D} = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{tA\,\cdot\,\nabla_x} D(U_o) e^{-tA\,\cdot\,\nabla_x}\,\mathrm{d}t\,.$$

Jiang-L show this system has a quadratic entropy dissipation, and that under mild assumptions (satisfied by the weakly compressible Navier-Stokes approximation) it has global solutions.

5. Survey of Some Results

The goals of the BGL program (1989) are (1) to identify those fluid dynamical systems that can be so derived, and (2) to give a full mathematical justification of those formal derivations.

The main result of [BGL93] for the Navier-Stokes limit is to recover the motion equation for a discrete-time version of the Boltzmann equation assuming the DiPerna-Lions solutions satisfy the local conservation of momentum and with the aid of a mild compactness assumption.

This result fell short of the goal in five respects.

- First, the heat equation was not treated because the $|v|^2 v$ terms in the heat flux could not be controlled.

- Second, local momentum conservation was assumed because DiPerna-Lions solutions are not known to satisfy the local conservation law of momentum (or energy) that one would formally expect.

- Third, unnatural technical assumptions were made on the Boltzmann collision kernel.

- Fourth, the discrete-time case was treated in order to avoid having to control the time regularity of the acoustic modes.

- Finally, a mild compactness assumption was required to pass to the limit in certain nonlinear terms.

In recent works all of these shortcomings have been overcome.

Review of the Scaling Relationships

Consider the scaled Boltzmann equation

$$\tau_\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} \mathcal{B}(F_\epsilon, F_\epsilon) \,,$$
$$F_\epsilon = M\Big(1 + \delta_\epsilon g_\epsilon\Big) \,.$$

One derives the acoustic system when $\tau_\epsilon = 1$ and

$$\delta_\epsilon \to 0 \,.$$

One derives the incompressible Stokes, the incompressible Navier-Stokes, and the incompressible Euler system when $\tau_\epsilon = \max\{\epsilon, \delta_\epsilon\}$, $\delta_\epsilon \to 0$ and respectively

$$\frac{\delta_\epsilon}{\epsilon} \to 0 \,, \qquad \frac{\delta_\epsilon}{\epsilon} \to 1 \,, \qquad \frac{\delta_\epsilon}{\epsilon} \to \infty \,.$$

Scaling of the Fluctuations

The scaling of the fluctuations is controlled by assuming that

$$\int \left\langle F_\epsilon^{in} \log\left(\frac{F_\epsilon^{in}}{M}\right) - F_\epsilon^{in} + M \right\rangle \mathrm{d}x < C^{in}\delta_\epsilon^2 \,.$$

The entropy inequality then implies that

$$F_\epsilon = M\Big(1 + \delta_\epsilon g_\epsilon\Big) \,,$$

where g_ϵ is compact in w-L^1. Moreover, it implies that every limit point must have the form of an infinitesimal Maxwellian

$$g_\epsilon \to \rho + v \cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\theta \,,$$

where ρ, u, θ are in L^2.

$\boxed{\textbf{Bardos-Golse-Levermore}}$

Bardos, Golse, and Levermore [BGL98] recover the acoustic and the Stokes limits for the Boltzmann equation for cutoff collision kernels that arise from Maxwell potentials. In doing so, they control the energy flux and *establish the local conservation laws of momentum and energy in the limit*. The scaling they used was not optimal, essentially requiring

$$\frac{\delta_\epsilon}{\epsilon} \longrightarrow 0 \quad \text{rather than} \quad \delta_\epsilon \longrightarrow 0 \quad \text{for the acoustic limit}\,,$$
$$\frac{\delta_\epsilon}{\epsilon^2} \longrightarrow 0 \quad \text{rather than} \quad \frac{\delta_\epsilon}{\epsilon} \longrightarrow 0 \quad \text{for the Stokes limit}\,.$$

Lions-Masmoudi

Lions and Masmoudi [LM00] recover the Navier-Stokes motion equation with the aid of only the local conservation of momentum assumption and the nonlinear compactness assumption that were made in [BGL93]. However, they do not recover the heat equation and they retain the same unnatural technical assumptions made in [BGL93] on the collision kernel.

There were two key new ingredients in their work. First, they were able to control the time regularity of the acoustic modes. Second, they were able to prove that the contribution of the acoustic modes to the limiting motion equation is just an extra gradiant term that can be incorporated into the pressure term.

$\boxed{\textbf{Lions-Masmoudi - 2}}$

They also recover the Stokes motion equation without the local conservation of momentum assumption and with essentially optimal scaling. However, they do not recover the heat equation and they retain the same unnatural technical assumptions made in [BGL93] on the collision kernel.

There are two reasons they do not recover the heat equation. First, it is unknown whether or not DiPerna-Lions solutions satisfy a local energy conservation law. Second, even if local energy conservation were assumed, the techniques they used to control the momentum flux would fail to control the heat flux.

Golse-Levermore - 1

Golse and Levermore [GL01] recover the acoustic and Stokes systems. They make natural assumptions on the collision kernel that include those classically derived from hard potentials.

For the Stokes limit they recover both the motion and heat equations with a near optimal scaling.

For the acoustic limit the scaling they used was not optimal, essentially requiring

$$\frac{\delta_\epsilon}{\epsilon^{\frac{1}{2}}} \longrightarrow 0 \qquad \text{rather than} \qquad \delta_\epsilon \longrightarrow 0\,.$$

Golse-Levermore - 2

There were two key new ingredients in this work. First, they control the local momentum and energy conservation defects of the DiPerna-Lions solutions with dissipation rate estimates that allowed them to recover these local conservation laws in the limit. Second, they also control the heat flux with dissipation rate estimates.

Because they treat the linear Stokes case, they do not face the need either to control the acoustic modes or for a compactness assumption, both of which are used to pass to the limit in the nonlinear terms in [LM00].

Saint Raymond

Without making any nonlinear compactness hypothesis, Saint Raymond [SR98] recovers the Navier-Stokes motion equation for the BGK model. This was a fundamental advance, but it took some time to extract the essential ingredients in a way that would impact the Boltzmann equation.

The flavor of her result is that every appropriately scaled family of BGK solutions has fluctuations that are compact and that every limit point of these fluctuations is an infinitesimal Maxwellian governed by the Navier-Stokes motion equation.

Golse-Saint Raymond

Without making any nonlinear compactness hypothesis, Golse-Saint Raymond [GSR04] recover the Navier-Stokes system for the Boltzmann equation with Grad-cutoff collision kernels that arise from Maxwell potentials. Their major breakthrough was the development of a new L^1 averaging lemma to prove the compactness assumption. This was extracted from Saint Raymond [SR98] where she recovered the Navier-Stokes limit for the BGK model. Their proof also employs key elements from [LM00, GL01] and from earlier work of Saint Raymond [SR99] on the incompressible Euler limit.

They have extended their result to the hard sphere collision kernel.

$\boxed{\textbf{Levermore-Masmoudi}}$

This extends the results of Golse and Saint Raymond. It recovers the Navier-Stokes-Fourier system for the Boltzmann equation with weakly cut-off collision kernels that arise from a wide range of hard and soft potentials.

Using the L^1 averaging lemma of Golse-Saint Raymond, they show that this nonlinear compactness hypothesis is satisfied. New estimates allow one to extend the analysis beyond Grad cutoff collision kernels.

These new estimates also allow one to carry out the acoustic and Stokes limits for soft potentials.

Jiang-Levermore

A relative entropy method is used to show (assuming a local energy conservation law) that over time scales on the order of $1/\epsilon$ one has

$$g_\epsilon \sim \rho_\epsilon + v \cdot u_\epsilon + \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right)\theta_\epsilon \,,$$

where ρ_ϵ, u_ϵ, and θ_ϵ solve the weakly compressible Stokes system. Recall that ϵ appears in that system. The key point here is that the convergence is strong. Earlier works on the incompressible Stokes scaling obtained strong convergence only for "well-prepared" initial data — that is, for initial data with no acoustic modes in the limit.

5. Some Open Problems:

- the acoustic limit with optimal scaling ($\delta_\epsilon \longrightarrow 0$);
- any limit for noncutoff collision kernels, which would require an extension of DiPerna-Lions theory;
- dominant-balance Stokes, Navier-Stokes, and Euler limits;
- uniform in time results (for example, for the weakly compressible Stokes and Navier-Stokes approximations);
- any results for “beyond Navier-Stokes” approximations.