# From Boltzmann Equations to Gas Dynamics: Introduction to Boltzmann Equations

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## FROM BOLTZMANN EQUATIONS TO GAS DYNAMICS

# I. INTRODUCTION TO BOLTZMANN EQUATIONS

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# **1. Historical Introduction**

Euler first published his equations of fluid dynamics in 1757. His equations embodied the then known conservation laws of mass and momentum (F = ma). While these equations were sufficient to govern the dynamics liquids, Euler realized that his theory was incomplete for gases. He knew that the pressure of a gas depended on its density and temperature (ideal gas law), but had no additional equation to govern this additional unknown.

Fourier introduced his heat equation around 1810. He viewed it as governing the dynamics of a massless heat-carrying substance called caloric that permeated all matter. His theory worked well in solids and some fluids.

By the early 1800's experimental evidence had made it clear that the Euler equations did not give a correct description of fluid dynamics even for liquids. In 1823 this led Navier, who was inspired by Fourier, to introduce an additional term into the momentum equation to model viscosity. In the 1840s Mayer, Joule, Helmholz, and others formulated the conservation law of energy when experiments indicated that heat was a form of energy. This became the first law of thermodynamics.

This led to a new unified theory of gas dynamics. The Euler equations were supplemented with an energy equation, thereby closing the system, while Navier's viscosity and Fourier's thermal conductivity terms were included in a consistent way. The resulting system became known as the Navier-Stokes-Fourier, or simply as the *Navier-Stokes system* of gas dynamics. It also respects the second law of thermodynamics, which was formulated in the 1850s.

When viscosity and thermal conductivity are neglected, the system is still called the *Euler system* of gas dynamics. This was the system studied by Riemann in 1851.

In founding kinetic theory, Maxwell (1860, 1866) and Boltzmann (1872) were testing the hypothesis that heat was the kinetic energy of molecules. Their theory led to formulas for the viscosity and thermal conductivity coefficients in terms of the Newtonian intermolecular force law. This is one of the earliest examples of what we now call an up-scaling theory.

Their theory was controversial at the time because the notion of molecule was not generally accepted, and because their arguments had many gaps — many of which still have to be bridged.

One fundamental question they had to address was how reversible microscopic Newtonian mechanics could lead to irreversible macroscopic dynamics. Maxwell introduced the famous "Maxwell demon" as a device to illustrate the statistical nature of macroscopic irreversibility in an 1867 private letter to Kelvin. The demon made its first public appearance in Maxwell's "Theory of Heat" published in 1871. Mathematicians came down on both sides of this controversy, which became very intense in the 1890's. Klein and Hilbert sided with Boltzmann. (Maxwell had died in 1879.) Poincare and Zermelo took the other side. They claimed that the Boltzmann equation is inconsistent with the Poincare Recurrence Theorem, which had appeared in 1890.

Experimental evidence for the existence of atoms and molecules (based on Einstein's 1905 theory of Brownian motion) became generally accepted around 1906. That changed, but did not stop the debate over the validity of the Boltzmann equation. (Poincare did switch sides.) It was only in the 1970's that the combination of careful experiments and careful simulations led to a general belief in its validity.

The Boltzmann equation has been mathematically justified (for finite time) only for the case of elastic hard spheres by Landford (1974) in the so-called Boltzmann-Grad limit.

Here we will assume the validity of the Boltzmann equation, and will use it as the starting point to establish fluid dynamical systems. Hilbert (1900 ICM) specifically called for this problem to be addressed as part of his sixth problem.

Maxwell's derivation of the Navier-Stokes system from the Boltzmann equation rests on arguments about how various terms in the "Boltzmann equation" balance each other. These balance arguments seemed arbitrary to some, so Hilbert (1912) proposed that such derivations should be based on a systematic expansion in a small nondimensional parameter, which we now call the Knudsen number. Later Enskog (1916) proposed a somewhat different systematic expansion, now often called the Chapman-Enskog expansion, in the same small parameter. Full justification of traditional compressible fluid dynamical approximations based on a formal Hilbert or Chapman-Enskog expansion has proven difficult because the basic well-posedness and regularity questions remain open for both the compressible fluid systems and the Boltzmann equation.

The problem is exacerbated by the fact that to bound the error of these asymptotic expansions requires control of successively higher order spatial derivatives of the fluid variables.

By the later half of the 20th century theories of local in time well-posedness for classical solutions had been developed by Grad, Ukai, Nishida, Caflisch, and many others. The resulting justifications thereby were restricted to a meager subset of all physically natural initial data and usually to finite times. In 1988 DiPerna-Lions gave a theory of global solutions to the Boltzmann equation for all physically natural initial data. This theory rests on a compactness argument, and therefore, does not yield uniqueness.

Their theory is similar in spirit to that of Leray (1934) regarding global solutions to the incompressible Navier-Stokes system for all physically natural initial data. It is natural to ask

"Does the Leray theory follow from that of DiPerna-Lions?"

It was shown by Golse-Saint Raymond in 2004 that the answer to this question is "yes". This result is part of a program that was begun in 1989 when Bardos-Golse-L gave a new formal derivation of the incompressible Navier-Stokes system from the Boltzmann equation. Rather than use traditional expansions, they gave a moment-based derivation, which puts fewer demands on the well-posedness and regularity theory.

More generally, this program seeks to:

- study derivations of linear or weakly nonlinear fluid dynamical systems, such as the acoustic system and the incompressible systems for which global theories exist;
- use moment-based formal derivations, which put fewer demands on the well-posedness and regularity theory;
- work within the framework of DiPerna-Lions solutions, thereby within the class of all physically natural initial data.

The second lecture will survey this program. The third lecture will present an extension of the Golse-Saint Raymond result.

#### **2. Boltzmann Equation Preliminaries**

The state of a fluid composed of identical point particles confined to a spatial domain  $\Omega \subset \mathbb{R}^D$  is described at the kinetic level by a mass density F over the single-particle phase space  $\mathbb{R}^D \times \Omega$ . More specifically,  $F(v, x, t) \, dv \, dx$  gives the mass of the particles that occupy any infinitesimal volume  $dv \, dx$  centered at the velocity  $v \in \mathbb{R}^D$  and the position  $x \in \Omega$  at the instant of time  $t \geq 0$ . To remove complications due to boundaries, we take  $\Omega$  to be the periodic domain  $\Omega = \mathbb{T}^D = \mathbb{R}^D / \mathbb{L}^D$ , where  $\mathbb{L}^D \subset \mathbb{R}^D$  is any D-dimensional lattice. Here  $D \geq 2$ .

The evolution of F = F(v, x, t) is governed by the Boltzmann equation:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \qquad F(v, x, 0) = F^{in}(v, x) \ge 0.$$
 (1)

The Boltzmann collision operator  $\mathcal{B}$  models binary collisions. It acts only on the v argument of F. It is formally given by

$$\mathcal{B}(F,F) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^{D}} (F_{1}'F' - F_{1}F) b(\omega, v_{1} - v) \,\mathrm{d}\omega \,\mathrm{d}v_{1} \,, \qquad (2)$$

where  $v_1$  ranges over  $\mathbb{R}^D$  endowed with its Lebesgue measure  $dv_1$  while  $\omega$  ranges over the unit sphere  $\mathbb{S}^{D-1} = \{\omega \in \mathbb{R}^D : |\omega| = 1\}$  endowed with its rotationally invariant measure  $d\omega$ . The  $F'_1$ , F',  $F_1$ , and F appearing in the integrand designate  $F(\cdot, x, t)$  evaluated at the velocities  $v'_1$ , v',  $v_1$ , and v respectively, where the primed velocities are defined by

$$v'_{1} = v_{1} - \omega \,\omega \cdot (v_{1} - v), \qquad v' = v + \omega \,\omega \cdot (v_{1} - v),$$
 (3)

for any given  $(\omega, v_1, v) \in \mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$ .

Quadratic operators like  $\mathcal{B}$  are extended by polarization to be bilinear and symmetric.

The unprimed and primed velocities are possible velocities for a pair of particles either before and after, or after and before, they interact through an elastic binary collision. Conservation of momentum and energy for particle pairs during collisions is expressed as

$$v + v_1 = v' + v'_1$$
,  $|v|^2 + |v_1|^2 = |v'|^2 + |v'_1|^2$ 

Equation (3) represents the general nontrivial solution of these D+1 equations for the 4D unknowns  $v'_1$ , v',  $v_1$ , and v in terms of the 3D-1 parameters  $(\omega, v_1, v)$ .

The collision kernel b is positive almost everywhere. The Galilean invariance of the collisional physics implies that b has the classical form

$$b(\omega, v_1 - v) = |v_1 - v| \Sigma(|\omega \cdot n|, |v_1 - v|),$$

where  $n = (v_1 - v)/|v_1 - v|$  and  $\Sigma$  is the specific differential cross-section.

Maxwell gave a recipe (1866) for the collision kernel in terms of the intermolecular potential. For hard spheres of mass m and radius  $r_o$  it yields

$$b(\omega, v_1 - v) = |\omega \cdot (v_1 - v)| \frac{(2r_o)^{D-1}}{2m}$$

For a repulsive intermolecular potential of the form  $c/r^k$  with  $k > 2\frac{D-1}{D+1}$  it yields

$$b(\omega, v_1 - v) = \hat{b}(\omega \cdot n) |v_1 - v|^{\beta}$$
 with  $\beta = 1 - 2\frac{D-1}{k}$ 

,

where  $n = (v_1 - v)/|v_1 - v|$  while  $\hat{b}(\omega \cdot n)$  is positive almost everywhere and has even symmetry in  $\omega \cdot n$ . The condition  $k > 2\frac{D-1}{D+1}$  is equivalent to  $\beta > -D$ , which insures that  $b(\omega, v_1 - v)$  is locally integrable with respect to  $v_1 - v$ .

The cases  $\beta > 0$ ,  $\beta = 0$ , and  $\beta < 0$  are called the "hard", "Maxwell", and "soft" potential cases.

The function  $\hat{b}(\omega \cdot n)$  derived by Maxwell's recipe (1866) for potentials of the form  $c/r^k$  has a singularity that is not locally integrable at  $\omega \cdot n = 0$ . This was not a problem for Maxwell because he used a weak form of the Boltzmann equation that regularizes this singularity. Hilbert (1912) avoided this problem by studying the Boltzmann equation for only the hard sphere case. Grad (1954) was able to extended some of Hilbert's analysis by introducing a small deflection cutoff that requires  $\hat{b}(\omega \cdot n)$  to vanish like  $|\omega \cdot n|$  as  $\omega \cdot n \to 0$  — the so-called *Grad cutoff* condition.

More generally one can assume that  $\hat{b}(\omega \cdot n)$  satisfies the less restrictive small deflection cutoff condition

$$\int_{\mathbb{S}^{D-1}}\widehat{b}(\omega\cdot n)\,\mathrm{d}\omega<\infty\,.$$

This so-called *weak cutoff* condition is required to have the gain and loss terms of the Boltzmann collision operator make sense separately.

#### **Conservation Properties**

Maxwell (1866) showed the collision operator has the following property related to the conservation laws of mass, momentum, and energy.

For every measurable  $\zeta$  the following are equivalent:

• 
$$\zeta \in \text{span}\{1, v_1, \cdots, v_D, \frac{1}{2}|v|^2\};$$

• 
$$\zeta'_1 + \zeta' - \zeta_1 - \zeta = 0$$
 for every  $(\omega, v_1, v)$ ;

• 
$$\langle \zeta \mathcal{B}(f,f) \rangle = 0$$
 for "every"  $f = f(v)$ .

Here we have introduced the notation  $\langle \cdot \rangle = \int \cdot dv$ .

#### **Local Conservation Laws**

If F is a classical solution of the Boltzmann equation then F satisfies local conservation laws of mass, momentum, and energy:

 $\partial_t \langle F \rangle + \nabla_x \cdot \langle v F \rangle = 0,$  $\partial_t \langle v F \rangle + \nabla_x \cdot \langle v \otimes v F \rangle = 0,$  $\partial_t \langle \frac{1}{2} |v|^2 F \rangle + \nabla_x \cdot \langle v \frac{1}{2} |v|^2 F \rangle = 0.$ 

## **Dissipation Properties**

Boltzmann's H-Theorem (1872) states that the collision operator has the following property related to the dissipation of entropy and equilibrium.

$$\langle \log(f) \mathcal{B}(f, f) \rangle = -\iiint \frac{1}{4} \log \left( \frac{f_1' f'}{f_1 f} \right) (f_1' f' - f_1 f) b d\omega dv_1 dv$$
  
 
$$\leq 0 \quad \text{for "every"} f = f(v) .$$

Moreover, for "every" f = f(v) the following are equivalent:

- $\langle \log(f) \mathcal{B}(f, f) \rangle = 0$ ,
- $\mathcal{B}(f,f)=0$ ,
- *f* is a Maxwellian, where ...

Maxwellians have the form

$$f = \mathcal{M}(v; \rho, u, \theta) = \frac{\rho}{(2\pi\theta)^{\frac{D}{2}}} \exp\left(-\frac{|v-u|^2}{2\theta}\right),$$

with  $\rho$ , u, and  $\theta$  given by

$$\rho = \langle f \rangle, \quad \rho u = \langle v f \rangle, \quad \rho \theta = \frac{1}{D} \langle |v - u|^2 f \rangle.$$

Here  $\rho$  is the mass density, u is the bulk velocity, and  $\theta = k_B T/m$  where  $k_B$  is Boltzmann's constant, T is the temperature of the gas, and m is the molecular mass.

In 1860 Maxwell argued these were the local equilibrium kinetic densities of gas dynamics. In 1866 he showed they were equilibria of what we now call the Boltzmann collision operator. Both times he gave an argument that they were the only such equilibria, but there were gaps in each argument. In 1872 Boltzmann filled the gap in Maxwell's second argument with his H-Theorem.

#### Local Entropy Dissipation Law

If F is a classical solution of the Boltzmann equation then F satisfies a local entropy dissipation law:

 $\partial_t \langle (F \log(F) - F) \rangle + \nabla_x \cdot \langle v (F \log(F) - F) \rangle$ =  $\langle \log(F) \mathcal{B}(F, F) \rangle$ =  $- \iiint \frac{1}{4} \log \left( \frac{F'_1 F'}{F_1 F} \right) (F'_1 F' - F_1 F) b d\omega dv_1 dv$  $\leq 0.$ 

# **3. Connections to Classical Gas Dynamics**

Fluid dynamical regimes are ones in which collisions dominate. (This will be made more precise in the next lecture.) The H-Theorem suggests that collisions will then drive F towards a so-called local Maxwellian — namely that

$$F(v, x, t) \approx \mathcal{M}(v; \rho(x, t), u(x, t), \theta(x, t)),$$

where  $\rho(x,t)$ , u(x,t), and  $\theta(x,t)$  are given by

$$\rho = \langle F \rangle, \quad \rho u = \langle v F \rangle, \quad \rho \theta = \frac{1}{D} \langle |v - u|^2 F \rangle.$$

A local Maxwellian is generally not a solution of the Boltzmann equation. However in fluid dynamical regimes the solution of the Boltzmann equation is almost a local Maxwellian.

## **Compressible Euler Approximation**

When the local Maxwellian approximation is placed into the local conservation laws one obtains the compressible Euler system of gas dynamics

 $\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$ 

$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x(\rho \theta) = 0,$$

$$\partial_t (\rho(\frac{1}{2}|u|^2 + \frac{D}{2}\theta)) + \nabla_x \cdot (\rho u(\frac{1}{2}|u|^2 + \frac{D+2}{2}\theta)) = 0.$$

Here the pressure satisfies the ideal gas law  $(p = \rho\theta)$  while the specific internal energy satisfies the polytropic  $\gamma$ -law with  $\gamma = \frac{D+2}{D}$  ( $\varepsilon = \frac{D}{2}\theta$ ). Euler systems that govern more general ideal gases can be derived from more complicated molecular models.

When the local Maxwellian approximation is placed into the local entropy dissipation law one obtains the relation

$$\partial_t \left( \rho \log \left( \frac{\rho}{\theta^{\frac{D}{2}}} \right) \right) + \nabla_x \cdot \left( \rho u \log \left( \frac{\rho}{\theta^{\frac{D}{2}}} \right) \right) \le 0.$$

This is the local entropy dissipation (production) law one expects from a local version of the second law of thermodynamics.

#### **Compressible Navier-Stokes Approximation**

Rather than present a derivation of the Navier-Stokes system based on either the Hilbert or the Chapman-Enskog expansion, here we present a simple balance argument similar in spirit to that used by Maxwell in 1866. Decompose F as

$$F(v, x, t) = \mathcal{M}(v; \rho(x, t), u(x, t), \theta(x, t)) + \widetilde{F}(v, x, t)$$

where  $\mathcal{M}(v; \rho, u, \theta)$  is the local Maxwellian with  $(\rho, u, \theta)$  determined by

$$\langle F \rangle = \rho, \qquad \langle v F \rangle = \rho u, \qquad \langle \frac{1}{2} | v - u |^2 F \rangle = \frac{D}{2} \rho \theta,$$

and  $\widetilde{F}$  is the deviation of F from  $\mathcal{M}$ . One sees that  $\widetilde{F}$  satisfies

$$\langle \tilde{F} \rangle = 0$$
,  $\langle v \tilde{F} \rangle = 0$ ,  $\langle \frac{1}{2} |v - u|^2 \tilde{F} \rangle = 0$ .

Placing this decomposition into the local conservation laws yields

$$\begin{split} \partial_t \rho + \nabla_{\!\!x} \cdot (\rho u) &= 0 \,, \\ \partial_t (\rho u) + \nabla_{\!\!x} \cdot (\rho u \otimes u) + \nabla_{\!\!x} (\rho \theta) + \nabla_{\!\!x} \cdot \widetilde{S} &= 0 \,, \\ \partial_t (\rho (\frac{1}{2}|u|^2 + \frac{D}{2}\theta)) + \nabla_{\!\!x} \cdot (\rho u (\frac{1}{2}|u|^2 + \frac{D+2}{2}\theta)) + \nabla_{\!\!x} \cdot (\widetilde{S}u + \widetilde{q}) &= 0 \,, \end{split}$$
where  $\widetilde{S}$  and  $\widetilde{q}$  are defined by

$$\widetilde{S} = \langle (v-u) \otimes (v-u) \widetilde{F} \rangle, \qquad \widetilde{q} = \langle (v-u) \frac{1}{2} | v-u |^2 \widetilde{F} \rangle.$$

These are the stress and heat flux respectively. They are the only terms that arise from  $\tilde{F}$ .

Fluid dynamical systems are obtained by making approximations for  $\tilde{F}$ . The compressible Euler system is obtained by setting  $\tilde{F} = 0$ . In order to obtain a better fluid dynamical system, one must find a better approximation for the deviation  $\tilde{F}$ . One can show that  $\tilde{F}$  satisfies the so-called deviation equation

$$\partial_t \tilde{F} + \tilde{\mathcal{P}} v \cdot \nabla_x \tilde{F} + \tilde{\mathcal{P}} v \cdot \nabla_x \mathcal{M} = \mathcal{B}(\mathcal{M} + \tilde{F}, \mathcal{M} + \tilde{F}),$$

where  $\widetilde{\mathcal{P}} = \mathcal{I} - \mathcal{P}$  and  $\mathcal{P}$  is the operator given by

$$\mathcal{P}G = \mathcal{M}\left[\frac{\langle G \rangle}{\rho} + \frac{(v-u) \cdot \langle (v-u) G \rangle}{\theta} + \left(\frac{|v-u|^2}{2\theta} - \frac{D}{2}\right) \left\langle \left(\frac{|v-u|^2}{D\theta} - 1\right) G \right\rangle \right].$$

One can show  $\mathcal{P}$  is a projection (i.e.  $\mathcal{P}^2 = \mathcal{P}$ ). Because  $\mathcal{P}\tilde{F} = 0$ , one sees  $\tilde{\mathcal{P}} = \mathcal{I} - \mathcal{P}$  is a projection onto the deviations from local equilibria. These are orthogonal projections in  $L^2(dv/\mathcal{M})$ .

The Navier-Stokes approximation is obtained by arguing that  $\tilde{F}$  is much smaller than  $\mathcal{M}$  and taking the dominant term on each side of the deviation equation. More specifically, we make the approximations

$$\partial_t \widetilde{F} + \widetilde{\mathcal{P}} v \cdot \nabla_{\!\! x} \widetilde{F} + \widetilde{\mathcal{P}} v \cdot \nabla_{\!\! x} \mathcal{M} \approx \widetilde{\mathcal{P}} v \cdot \nabla_{\!\! x} \mathcal{M} ,$$

$$\mathcal{B}(\mathcal{M}+\widetilde{F},\mathcal{M}+\widetilde{F})=2\mathcal{B}(\mathcal{M},\widetilde{F})+\mathcal{B}(\widetilde{F},\widetilde{F})\approx 2\mathcal{B}(\mathcal{M},\widetilde{F})$$

We thereby argue that  $\widetilde{F}\approx\widetilde{F}_{\!NS}$  where  $\widetilde{F}_{\!NS}$  satisfies

$$\widetilde{\mathcal{P}}v\cdot\nabla_{x}\mathcal{M}=2\mathcal{B}(\mathcal{M},\widetilde{F}_{NS}).$$

This is the Navier-Stokes balance relation. It leads to the compressible Navier-Stokes system, although at this stage it may not be obvious how.

The left-hand side of the Navier-Stokes balance relation has the form

$$\widetilde{\mathcal{P}}v\cdot\nabla_{x}\mathcal{M} = \mathcal{M}\left(A\left(\frac{v-u}{\theta^{\frac{1}{2}}}\right):\nabla_{x}u + B\left(\frac{v-u}{\theta^{\frac{1}{2}}}\right)\cdot\frac{\nabla_{x}\theta}{\theta^{\frac{1}{2}}}\right),$$

where the non-dimensional functions A(v) and B(v) are defined by

$$A(v) = v \otimes v - \frac{1}{D} |v|^2 I$$
,  $B(v) = \frac{1}{2} |v|^2 v - \frac{D+2}{2} v$ .

Notice that A(v) is a traceless, symmetric matrix, while B(v) is a vector. The solution of the Navier-Stokes balance relation is

$$\widetilde{F}_{NS} = -\frac{\mathcal{M}}{\rho} \left( \widehat{A}(v-u;\theta) : \nabla_{x}u + \widehat{B}(v-u;\theta) \cdot \frac{\nabla_{x}\theta}{\theta^{\frac{1}{2}}} \right),$$

provided that  $\widehat{A}(v; \theta)$  and  $\widehat{B}(v; \theta)$  satisfy

$$-\frac{2}{M}\mathcal{B}(M, M\widehat{A}) = A(v/\theta^{\frac{1}{2}}), \qquad M\widehat{A} \perp 1, v, |v|^2,$$
$$-\frac{2}{M}\mathcal{B}(M, M\widehat{B}) = B(v/\theta^{\frac{1}{2}}), \qquad M\widehat{B} \perp 1, v, |v|^2,$$

with  $M = \mathcal{M}(v; 1, 0, \theta)$ . The linear operator on the left-hand side above is symmetric and nonnegative definite in  $L^2(M dv)$ .

Maxwell (1866) found explicit solutions to these equations for the case of so-called Maxwell molecules,  $\beta = 0$ . Hilbert (1912) showed that these equations have solutions for the hard-sphere case. More generally, he showed the linear operator above satisfies a Fredholm alternative. This kind of result has now been extended to all classical collision kernels such that  $\beta > -D$  and  $\hat{b}(\omega \cdot n)$  satisfies the weak cutoff condition. One can show that  $\hat{A}(v;\theta) = \tau_A(v;\theta)A(v/\theta^{\frac{1}{2}})$  and  $\hat{B}(v;\theta) = \tau_B(v;\theta)B(v/\theta^{\frac{1}{2}})$ .

Finally, the stress and heat flux can then be shown to have the form

$$\widetilde{S}_{NS} = -\mu(\theta) \Big[ \nabla_{x} u + (\nabla_{x} u)^{T} - \frac{2}{D} \nabla_{x} \cdot u I \Big],$$
  
$$\widetilde{q}_{NS} = -\kappa(\theta) \nabla_{x} \theta,$$

where  $\mu(\theta)$  and  $\kappa(\theta)$  are positive functions of  $\theta$  given by

$$\mu(\theta) = \frac{1}{(D+2)(D-1)} \theta \left\langle \mathcal{M}(v; 1, 0, \theta) \left| A\left(\frac{v}{\theta^{\frac{1}{2}}}\right) \right|^2 \tau_A(v; \theta) \right\rangle,$$
  
$$\kappa(\theta) = \frac{1}{D} \theta \left\langle \mathcal{M}(v; 1, 0, \theta) \left| B\left(\frac{v}{\theta^{\frac{1}{2}}}\right) \right|^2 \tau_B(v; \theta) \right\rangle.$$

One immediately sees that  $\mu(\theta)$  and  $\kappa(\theta)$  are the viscosity and thermal conductivity coefficients respectively. These functions of  $\theta$  are the only things in the Navier-Stokes system that depend on the collision kernel *b*, and therefore the only things that depend on details of the microsopic dynamics. The fact that they are independent of  $\rho$  was an important early prediction of kinetic theory that was subsequently confirmed by experiment.